On the Hyperbolicity of Small-World and Tree-Like Random Graphs

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Abstract

Hyperbolicity is a property of a graph that may be viewed as being a "soft" version of a tree, and recent empirical and theoretical work has suggested that many graphs arising in Internet and related data applications have hyperbolic properties. Here, we consider Gromov's notion of δ-hyperbolicity, and we establish several positive and negative results for small-world and tree-like random graph models. In particular, we show that small-world random graphs built from underlying grid structures do not have strong improvement in hyperbolicity, even when the rewiring greatly improves decentralized navigation. On the other hand, for a class of tree-like graphs called ringed trees that have constant hyperbolicity, adding random links among the leaves in a manner similar to the small-world graph constructions may easily destroy the hyperbolicity of the graphs, except for a class of random edges added using an exponentially decaying probability function based on the ring distance among the leaves. Our study provides the first significant analytical results on the hyperbolicity of a rich class of random graphs, which shed light on the relationship between hyperbolicity and navigability of random graphs, as well as on the sensitivity of hyperbolic δ to noises in random graphs.

Keywords: Complex networks, graph hyperbolicity, small-world networks, decentralized navigation

1 Introduction

Hyperbolicity, a property of metric spaces that generalizes the idea of Riemannian manifolds with negative curvature, has received considerable attention in both mathematics and computer science. When applied to graphs, one may think of hyperbolicity as characterizing a "soft" version of a tree—trees have hyperbolicity zero, and graphs that "look like" trees in terms of their metric structure have "small" hyperbolicity. Since trees are an important class of graphs and since tree-like graphs arise in numerous applications, the idea of hyperbolicity has received attention in a range of applications. For example, it has found usefulness in the visualization of the Internet, the Web, and other large graphs [30, 31, 36, 35, 42]; it has been applied to questions of compact routing, navigation, and decentralized search in Internet graphs and small-world social networks [13, 10, 25, 1, 26, 5, 27]; and it has been applied to a range of other problems such as distance estimation, network security, sensor networks, and traffic flow and congestion minimization [2, 17, 18, 19, 37, 12].

The hyperbolicity of graphs are typically measured by Gromov's hyperbolic δ [15, 7] (see Section 2). The hyperbolic δ of a graph measures the "tree-likeness" of the graph in terms of the graph distance metric. It can range from 0 up to the half of the graph diameter, with trees having $\delta = 0$, while "circle graphs" and "grid graphs" having large δ equal to roughly half of their diameters.

In this paper, we study the δ-hyperbolicity of families of random graphs that intuitively have some sort of tree-like or hierarchical structure. Our motivation comes from two angles. First, although there are a number of empirical studies on the hyperbolicity of real-world and random graphs [2, 17, 33, 32, 37, 12], there are essentially no systematic analytical study on the hyperbolicity of popular random graphs. Thus, our work is intended to fill this gap. Second, a number of algorithmic studies show that good graph hyperbolicity leads to efficient distance labeling and routing schemes [8, 13, 11, 9, 28, 10], and the routing infrastructure of the Internet is also empirically shown to be hyperbolic [2]. Thus, it is interesting to further investigate if efficient routing capability implies good graph hyperbolicity.

To achieve our goal, we first provide fine-grained characterization of δ-hyperbolicity of graph families relative to the graph diameter: A family of random graphs is (a) *constantly hyperbolic* if their hyperbolic δ's are constant, regardless of the size or diameter of the graphs; (b) *logarithmically (or polylogarithmically) hyperbolic* if their hyperbolic δ's are in the order of logarithm (or polylogarithm) of the graph diameters; (c) *weakly hyperbolic* if their hyperbolic δ's grow asymptotically slower than the graph diameters; and (d) *not hyperbolic* if their hyperbolic δ's are at the same order as the graph diameters.

We study two families of random graphs. The first family is Kleinberg's grid-based small-world random graphs [22], which build random long-range edges among pairs of nodes with probability inverse proportional to the γ-th power of the grid distance of the pairs. Kleinberg shows that when γ equals to the grid dimension d, decentralized routing can be improved from $\Theta(n)$ in grid to $O(\mathrm{polylog}(n))$, where n is the number of vertices in the graph. Contrary to the improvement in decentralized routing, we show that when $\gamma = d$, with high probability the small-world graph is not polylogarithmically hyperbolic. We further show that when $0 \le \gamma < d$, the random small-world graphs is not hyperbolic and when $\gamma > 3$ and $d = 1$, the random graphs is not polylogarithmically hyperbolic. Although there still exists a gap between hyperbolic δ and graph diameter at the sweetspot of $\gamma = d$, our results already indicate that long-range edges that enable efficient navigation do not significantly improve the hyperbolicity of the graphs.

The second family of graphs is random *ringed trees*. A ringed tree is a binary tree with nodes in each level of the tree connected by a ring (Figure 1(d)). Ringed trees can be viewed as an idealized version of hierarchical structure with local peer connections, such as the Internet autonomous system (AS) topology. We show that ringed tree is quasi-isometric to the Poincaré disk, the well known hyperbolic space representation, and thus it is constantly hyperbolic. We then study how random additions of long-range links on the leaves of a ringed tree affect the hyperbolicity of random ringed trees. Note that due to the tree base structure, random ringed trees allow efficient routing within $O(\log n)$ steps using tree branches. Our results show that if the random long-range edges between leaves are added according to a probability function that decreases exponentially fast with the ring distance between leaves, then the resulting random graph is logarithmically hyperbolic, but if the probability function decreases only as a power-law with ring distance, or based on another tree distance measure similar to [23], the resulting random graph is not hyperbolic. Furthermore, if we use binary trees instead of ringed trees as base graphs, none of the above versions is hyperbolic. Taken together, our results indicate that δ-hyperbolicity of the graphs are quite sensitive to both base graph structures and probabilities of long-range connections.

To summarize, we provide the first significant analytical results on the hyperbolicity properties of important families of random graphs. Our results demonstrate that efficient routing performance does not necessarily mean good graph hyperbolicity (such as logarithmic hyperbolicity).

1.1 Related work

There has been a lot of work on search and decentralized search subsequent to Kleinberg's original work [22, 23], much of which has been summarized in the review [24]. In a parallel with this, there has been empirical and theoretical work on hyperbolicity of real-world complex networks as well as simple random graph models. On the empirical side, [2] showed that measurements of the Internet are negatively curved; [17, 18, 19, 33, 32] provided empirical evidence that randomized scale-free and Internet graphs are more hyperbolic than other types of random graph models; [37] measured the average δ and related curvature to congestion; and [12] measured treewidth and hyperbolicity properties of the Internet. On the theoretical side, one has [40, 19, 4, 38]. The most related to our results, and to our knowledge the only prior work to obtain significant results for the hyperbolicity of a class of random graph models, is [38], which proves that with positive probability extremely sparse Erdős-Rényi random graphs are not δ-hyperbolic for any positive constant δ.

More generally, we see two approaches connecting hyperbolicity with efficient routing in graphs. One approach study efficient computation of graph properties, such as diameters, centers, approximating trees, and packings and coverings for low hyperbolic-δ graphs and metric spaces [9, 8, 13, 10, 11]. In large part, the reason for this interest is that there are often direct consequences for navigation and routing in these graphs [13, 10, 25, 1]. While these results are of interest for general low hyperbolic-δ graphs, they can be less interesting when applied to small-world and other low-diameter random models of complex networks. To take one example, [9] provides a simple construction of a distance approximating tree for δ-hyperbolic graphs on n vertices; but the $O(\log n)$ additive-error guarantee is clearly less interesting for models in which the diameter of the graph is $O(\log n)$. Unfortunately, this $O(\log n)$ arises for a very natural reason in the analysis, and it is nontrivial to improve it for popular tree-like complex network models.

Another approach taken by several recent papers assumes a random graph model with small hyperbolicity and then shows that such random graphs lead to several common properties of small-world complex networks, including good navigability properties [5, 27, 28, 29]. While assuming low hyperbolicity in these studies makes intuitive sense, it is difficult to prove nontrivial results for Gromov's δ even for simple random graph models that are intuitively tree-like.

Understanding the relationship between these two approaches was one of the original motivations of our research. In particular, the difficulties in the above two approaches lead us to study hyperbolicity of small-world and tree-like random graphs.

Finally, ideas related to hyperbolicity have been applied in numerous other networks applications, e.g., to problems such as distance estimation, network security, sensor networks, and traffic flow and congestion minimization [41, 20, 21, 18, 37, 3], as well as large-scale data visualization [35]. The latter applications typically take important advantage of the idea that data are often hierarchical or tree-like and that there is "more room" in hyperbolic spaces of a given dimension than corresponding Euclidean spaces.

Paper organization. In Section 2 we provide basic concepts and terminologies on hyperbolic spaces and graphs that are needed in this paper. In Sections 3 and 4 we study the hyperbolicity of small-world random graphs and ringed tree based random graphs. For ease of reading, in each of Sections 3 and 4 we first summarize our technical results together with their implications (Sections 3.1 and 4.1), then provide the outline of the analyses (Sections 3.2 and 4.2), followed by the detailed technical proofs (Sections 3.3 and 4.3), and finally discuss extensions of our results to other related models (Sections 3.4 and 4.4). We discuss open problems and future directions related to our study in Section 5.

2 Preliminaries on hyperbolic spaces and graphs

Here, we provide basic concepts concerning hyperbolic spaces and graphs used in this paper; for more comprehensive coverage on hyperbolic spaces, see, e.g., [7].

2.1 Gromov's δ-hyperbolicity

In [15], Gromov defined a notion of hyperbolic metric space; and he then defined hyperbolic groups to be finitely generated groups with a Cayley graph that is hyperbolic. There are several equivalent definitions (up to a multiplicative constant) of Gromov's hyperbolic metric space [6]. In this paper, we will mainly use the following.

Definition 1 (Gromov's four-point condition)**.** *In a metric space (X, d), given u, v, w, x with $d(u,v) + d(w,x) \geq d(u,x) + d(w,v) \geq d(u,w) + d(v,x)$ in X, we note $\delta(u,v,w,x) = (d(u,v) + d(w,x) - d(u,x) - d(w,v))/2$. (X,d) is called* δ-hyperbolic *for some non-negative real number δ if for any four points $u,v,w,x \in X$, $\delta(u,v,w,x) \leq \delta$. Let $\delta(X,d)$ be the smallest possible value of such δ, which can also be defined as $\delta(X,d) = \sup_{u,v,w,x \in X} \delta(u,v,w,x)$.*

Given an undirected, unweighted and connected graph $G = (V, E)$, one can view it as a metric space (V, d_G), where $d_G(u,v)$ denotes the (geodesic) graph distance between two vertices u and v. Then, one can apply the above four point condition to define its δ-hyperbolicity, which we denote $\delta = \delta(G) = \delta(V, d_G)$ (and which we sometimes refer to simply as the hyperbolicity or the δ of the graph). Trees are 0-hyperbolic; and 0-hyperbolic graphs are exactly clique trees (or called block graphs), which can be viewed as cliques connected in a tree fashion [16]. Thus, it is often helpful to view graphs with a low hyperbolic δ as "thickened" trees, or in other words, as tree-like when viewed at large size scales.

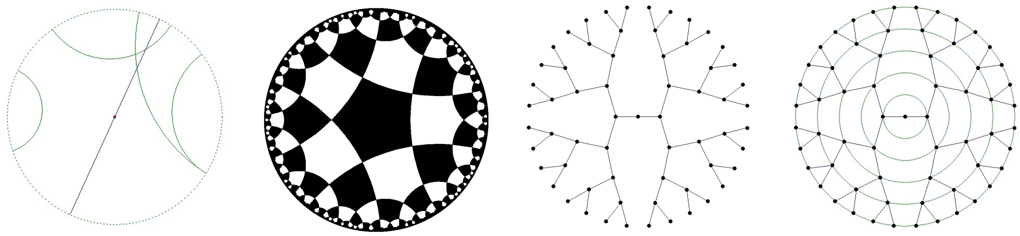
If we let $D(G)$ denote the diameter of the graph G, then, by the triangle inequality, we have $\delta(G) \leq D(G)/2$. We will use the asymptotic difference between the hyperbolicity $\delta(G)$ and the diameter $D(G)$ to characterize the hyperbolicity of the graph G.

Definition 2 (Hyperbolicity of a graph)**.** *For a family of graphs $\mathcal{G}$ with diameter $D(G), G \in \mathcal{G}$ going to infinity, we say that graph family $\mathcal{G}$ is* constantly (resp. logarithmically, polylogarithmically, or weakly) hyperbolic*, if $\delta(G) = O(1)$ (resp. $O(\log D(G))$, $O((\log D(G))^c)$ for some constant $c > 0$, or $o(D(G))$) when $D(G)$ goes to infinity; and $\mathcal{G}$ is* not hyperbolic *if $\delta(G) = \Theta(D(G))$, where $G \in \mathcal{G}$.*

The above definition provides more fine-grained characterization of hyperbolicity of graph families than one typically sees in the literature, which only discusses whether or not a graph family is constantly hyperbolic.

2.2 Rips condition

Rips condition [15, 7] is a technically equivalent condition to the Gromov's four point condition up to a constant factor. We use the Rips condition when analyzing the δ-hyperbolicity of ringed trees. In a metric space (X, d), we define a *geodesic segment* $[u, v]$ between two points u, v to be the image of a function $\rho : [0, d(u,v)] \to [u,v]$ satisfying $\rho(0) = u$, $\rho(d(u,v)) = v$, $d(\rho(s), \rho(t)) = |s-t|$ for any $s, t \in [0, d(u,v)]$. We say that a metric space is *geodesic* if every pair of its points has a geodesic segment, not necessarily unique. In a geodesic metric space (X,d), given u, v, w in X, we denote $\Delta(u,v,w) = [u,v] \cup [v,w] \cup [w,u]$ a *geodesic triangle*. $[u,v]$, $[v,w]$, $[w,u]$ are called *sides* of $\Delta(u,v,w)$. We should note that, in general, geodesic segments and geodesic triangles are not unique up to their endpoints.



(a) Poincaré disk (b) Tessellation of Poincaré disk (c) Binary tree (d) Ringed tree

Figure 1: Poincaré disk, its tessellation, a binary tree, and a ringed tree.

In a metric space, it is sometimes convenient to consider distances between point sets in the following way. We say that a set S is within distance d to another set T if S is contained in the ball $B(T, d)$ of all points within distance d to some point in T. We say that S and T are within distance d to each other if S is within distance d to T and vice versa.

Definition 3 (Rips condition)**.** *A geodesic triangle $\Delta(u, v, w)$ in a geodesic metric space (X, d) is called* δ-slim *for some non-negative real number δ if any point on a side is within distance δ to the union of the other two sides. (X, d) is called* Rips δ-hyperbolic *if every geodesic triangle in (X, d) is δ-slim. We denote $\delta_{Rips}(X, d)$ the smallest possible value of such δ (could be infinity).*

It is known (see, e.g., [14, 7, 9]) that $\delta(X, d)$ and $\delta_{Rips}(X, d)$ differ only within a multiplicative constant. In particular, $\delta(X, d) \leq 8\delta_{Rips}(X, d)$ and $\delta_{Rips}(X, d) \leq 4\delta(X, d)$. Since we are only concerned with asymptotic growth of $\delta(X, d)$, Rips condition can be used in place of the Gromov's four point condition.

For an undirected unweighted graph $G = (V, E)$, we can also treat it as a geodesic metric space with every edge interpreted as a segment of length 1, and thus use the Rips condition to define its hyperbolicity, which we denote as $\delta_{Rips}(G)$. Note that in the case of unweighted graph, when considering the distance between two geodesics on the graph, we only consider the distance among the vertices, since other points on the edges can add at most 2 to the distance between vertices.

2.3 Poincaré disk

The Poincaré disk (see Figure 1(a) for an illustration) is a well-studied hyperbolic metric space. Although in this paper we touch upon it only briefly when we study ringed-tree graphs, it is useful to convey intuition about hyperbolicity and tree-like behavior.

Definition 4. *Let $D = B(0, 1)$ be a open disk on the complex plane with origin 0 and radius 1, with the following distance function:*

$$d(u, v) = \operatorname{arccosh}\left(1 + \frac{2\|u - v\|^2}{(1 - \|u\|^2)(1 - \|v\|^2)}\right).$$

(D, d) is a metric space. We call it the Poincaré disk.

Visually, a (hyperbolic) line in the Poincaré disk is the segment of a circle in the disk that is perpendicular to the circular boundary of the disk, and thus all lines bend inward towards the origin. The hyperbolic distance

between two points in the disk of fixed distance in the complex plane increase exponentially fast when they moves towards the boundary of the disk, meaning that there is much more "space" towards the boundary than around the origin. This can be seen from a tessellation of the Poincaré disk, as shown in Figure 1(b).

2.4 Quasi-isometry

Quasi-isometry, defined as follows, is a concept used to capture the large-scale similarity between two metric spaces.

Definition 5 (Quasi-isometry)**.** *For two metric spaces $(X, d_X), (Y, d_Y)$, we say that $f : X \to Y$ is a (λ, ϵ)-quasi-isometric embedding from X to Y if for any $u, v \in X$,*

$$\frac{1}{\lambda} d_X(u, v) - \epsilon \leq d_Y(f(u), f(v)) \leq \lambda d_X(u, v) + \epsilon.$$

Furthermore, if the ϵ neighborhood of $f(Y)$ covers X, then we say that f is a (λ, ϵ)-quasi-isometry. Moreover, we say that X, Y are quasi-isometric if such a (λ, ϵ)-quasi-isometry exists for some constants λ and ϵ.

If two metric spaces are quasi-isometric with some constant, then they have the same "large-scale" behavior. For example, the d-dimensional grid $\mathbb{Z}^d$ and the d-dimensional Euclidean space $\mathbb{R}^d$ are quasi-isometric, realized by the $(\sqrt{d}, \sqrt{d}/2)$-quasi-isometric embedding $(x, y) \mapsto (x, y)$. As a second example, consider an infinite ringed-tree: start with a binary tree (illustrated in Figure 1(c)) and then connect all vertices at a given tree level into a ring. This is defined more formally in Section 4, but an example is illustrated in Figure 1(d). As we prove in Section 4, the infinite ringed tree is quasi-isometric to the Poincaré disk—thus it may be equivalently viewed as a "softened" binary tree or as a "coarsened" Poincaré disk.

Quasi-isometric embeddings have the important property of preserving hyperbolicity, up to a constant factor, as given by the following proposition.

Proposition 1 (Theorem 1.9, Chapter III.H of [7])**.** *Let X and X' be two metric spaces and let $f : X' \to X$ be a (λ, ϵ)-quasi-isometric embedding. If X is δ-hyperbolic, then X' is δ'-hyperbolic, where δ' is a function of δ, λ, and ϵ.*

3 δ-hyperbolicity of grid-based small-world graphs

In this section, we consider the δ-hyperbolicity of graphs constructed according to the small-world graph model as formulated by Kleinberg [22], in which long-range edges are added on top of a base grid, which is a discretization of a low-dimensional Euclidean space.

The model starts with n vertices forming a d-dimensional base grid (with wrap-around). More precisely, given positive integers n and d such that $n^{1/d}$ is also an integer, let $B = (V, E)$ be the base grid, with $V = \{(x_1, x_2, \ldots, x_d) \mid x_i \in \{0, 1, \ldots, n^{1/d} - 1\}, i \in [d]\}$, $E = \{((x_1, x_2, \ldots, x_d), (y_1, y_2, \ldots, y_d)) \mid \exists j \in [d], y_j = x_j + 1 \mod n^{1/d} \text{ or } y_j = x_j - 1 \mod n^{1/d}, \forall i \neq j, y_i = x_i\}$. Let d_B denote the graph distance metric on the base grid B. We then build a random graph G on top of B, such that G contains all vertices and all edges (referred to as grid edges) of B, and for each node $u \in V$, it has one long-range edge (undirected) connected to some node $v \in V$, with probability proportional to $1/d_B(u, v)^\gamma$, where $\gamma \geq 0$ is a parameter. We refer to the probability space of these random graphs as $KSW(n, d, \gamma)$; and we let $\delta(KSW(n, d, \gamma))$ denote the random variable of the hyperbolic δ of a randomly picked graph G in $KSW(n, d, \gamma)$. Recall that Kleinberg showed that the small-world graphs with $\gamma = d$ allow efficient

decentralized routing (with $O(\log^2 n)$ routing hops in expectation), whereas graphs with $\gamma \neq d$ do not allow any efficient decentralized routing (with $\Omega(n^c)$ routing hops for some constant c) [22]; and note that the base grid B has large hyperbolic δ, i.e., $\delta(B) = \Theta(n^{1/d}) = \Theta(D(B))$. Intuitively, the structural reason for the efficient routing performance at $\gamma = d$ is that long-range edges are added "hierarchically" such that each node's long-range edges are nearly uniformly distributed over all "distance scales".

3.1 Results and their implications

The following theorem summarizes our main technical results on the hyperbolicity of small-world graphs for different combinations of d and γ.

Theorem 1. *With probability $1 - o(1)$ (when n goes to infinity), we have*

1. $\delta(KSW(n, d, \gamma)) = \Omega((\log n)^{\frac{1}{1.5(d+1)+\varepsilon}})$ *when $d \geq 1$ and $\gamma = d$, for any $\varepsilon > 0$ independent of n;*
2. $\delta(KSW(n, d, \gamma)) = \Omega(\log n)$ *when $d \geq 1$ and $0 \leq \gamma < d$; and*
3. $\delta(KSW(n, d, \gamma)) = \Omega(n^{\frac{\gamma-2}{\gamma-1}-\epsilon})$ *when $d = 1$ and $\gamma > 3$, for any $\epsilon > 0$ independent of n.*

This theorem, together with the results of [22] on the navigability of small-world graphs, have several implications. The first result shows that when $\gamma = d$, with high probability the hyperbolic δ of the small-world graphs is at least $c(\log n)^{\frac{1}{1.5(d+1)}}$ for some constant c. We know that the diameter is $\Theta(\log n)$ in expectation when $\gamma = d$ [34]. Thus the small-world graphs at the sweetspot for efficient routing is not polylogarithmically hyperbolic, i.e., δ is not $O(\log^c \log n)$-hyperbolic for any constant $c > 0$. However, there is still a gap between our lower bound the upper bound provided by the diameter, and thus it is still open whether small-world graphs are weakly hyperbolic or not hyperbolic. Overall, though, our result indicates no drastic improvement on the hyperbolicity (relative to the improvement of the diameter) for small-world graphs at the sweetspot (where a dramatic improvement was obtained for the efficiency of decentralized routing).

The second result shows that when $\gamma < d$, then $\delta = \Omega(\log n)$. The diameter of the graph in this case is $\Theta(\log n)$ [34]; thus, we see that when $\gamma < d$ the hyperbolic δ is asymptotically the same as the diameter, i.e., although δ decreases as edges are added, small-world graphs in this range are not hyperbolic. The third result concerns the case $\gamma > d$, in which case the random graph degenerates towards the base grid (in the sense that most of all of the long-range edges are very local), which itself is not hyperbolic. For the general γ, we show that for the case of $d = 1$ the hyperbolic δ is lower bounded by a (low-degree) polynomial of n; this also implies that the graphs in this range are not polylogarithmically hyperbolic. Note that our polynomial exponent $\frac{\gamma-2}{\gamma-1} - \epsilon$ matches the diameter lower bound proven in [39].

3.2 Outline of the proof of Theorem 1

In this subsection, we provide a summary of the proof of Theorem 1. In our analysis, we use two different techniques, one for the first two results in Theorem 1, and the other for the last result; in addition, for the first two results, we further divide the analysis into the two cases $d \geq 2$ and $d = 1$.

When $d \geq 2$ and $0 \leq \gamma \leq d$, the main idea of the proof is to pick a square grid of size ℓ_0 (it does not matter in which dimension the square is picked from). We know that when only grid distance is considered, the four corners of the square grid have the Gromov δ value equal to ℓ_0. We will show that, as long as ℓ_0 is not very large (to be exact, $O((\log n)^{\frac{1}{1.5(d+1)+\varepsilon}})$ when $\gamma = d$ and $O(\log n)$ when $0 \leq \gamma < d$), the probability

that any pair of vertices on this square grid have a shortest path shorter than their grid distance after adding long-range edges is close to zero (as n tends to infinity). Therefore, with high probability, the four corners selected have Gromov δ as desired in the lower bound results.

To prove this result, we study the probability that any pair of vertices u and v at grid distance ℓ are connected with a path that contains at least one long-range edge and has length at most ℓ. We upper bound such ℓ's so that this probability is close to zero. To do so, we first classify such paths into a number of categories, based on the pattern of paths connecting u and v: how it alternates between grid edges and long-range edges, and the direction on each dimension of the grid edges and long-range edges (i.e., whether it is the same direction as from u to v in this dimension, or the opposite direction, or no move in this dimension). We then bound the probability of existing a path in each category and finally bound all such paths in aggregate. The most difficult part of the analysis is the bounding of the probability of existing a path in each category.

For the case of $d = 1$ and $0 \leq \gamma \leq d$, the general idea is similar to the above. The difference is that we do not have a base square to start with. Instead, we find a base ring of length $\Theta(\ell_0)$ using one long-range edges e_0, where ℓ_0 is fixed to be the same as the case of $d \geq 2$. We show that with high probability, (a) such an edge e_0 exists, and (b) the distance of any two vertices on the ring is simply their ring distance. This is enough to show the lower bound on the hyperbolic δ.

For the case of $\gamma > 3$ and $d = 1$, a different technique is used to prove the lower bound on hyperbolic δ. We first show that, in this case, with high probability all long-range edges only connect two vertices with ring distance at most some $\ell_0 = o(\sqrt{n})$. Next, on the one dimensional ring, we first find two vertices A and B at the two opposite ends on the ring. Then we argue that there must be a path $\mathcal{P}^+_{AB}$ that only goes through the clockwise side of ring from A to B, while another path $\mathcal{P}^-_{AB}$ that only goes through the counter-clockwise side of the ring from A to B, and importantly, the shorter length of these two paths are at most $O(\ell_0)$ longer than the distance between A and B. We then pick the middle point C and D of $\mathcal{P}^+_{AB}$ and $\mathcal{P}^-_{AB}$, respectively, and argue that the δ value of the four points A, B, C, and D give the desired lower bound.

3.3 Detailed proof of Theorem 1

3.3.1 The case of $d \geq 2$ and $0 \leq \gamma \leq d$

For this case, let $n' = n^{1/d}$ be the number of vertices on one side of the grid. For convenience, our main analysis for this case uses n' instead of n.

Lemmas for calculation. We first provide a couple of lemmas used in our probability calculation.

Lemma 1. *There exists a constant c_1, such that for any $k, m \in \mathbb{Z}^+$, we have*

$$\sum_{\substack{y_1+\cdots+y_k=m \\ y_1,\ldots,y_k\in\mathbb{Z}^+}} \frac{1}{y_1 y_2 \cdots y_k} \leq \frac{(c_1 \ln m)^{k-1}}{m},$$

where the left side is considered to be 0 for $k > m$; and for $k = m = 1$, the right side 0^0 is considered to be 1.

Proof. For $k = 1$, it is trivial. For $k = 2$, we have

$$\sum_{\substack{y_1+y_2=m \\ y_1,y_2\in\mathbb{Z}^+}} \frac{1}{y_1y_2} \le 2\left(\frac{1}{\lfloor m/2\rfloor\cdot\lceil m/2\rceil}+\cdots+\frac{1}{(m-1)\cdot 1}\right) \le \frac{2}{\lfloor m/2\rfloor}\left(\frac{1}{\lceil m/2\rceil}+\cdots+\frac{1}{1}\right)$$
$$< c_1\frac{\ln m}{m},$$

where c_1 is roughly 4.

Suppose the lemma holds for $k-1$, with $k \ge 3$. The induction hypothesis is

$$\sum_{\substack{y_1+\cdots+y_{k-1}=m \\ y_1,\ldots,y_{k-1}\in\mathbb{Z}^+}} \frac{1}{y_1y_2\cdots y_{k-1}} \le \frac{(c_1\ln m)^{k-2}}{m},$$

Since the logarithm function is increasing, we have

$$\begin{aligned}\sum_{\substack{y_1+\cdots+y_k=m \\ y_1,\ldots,y_k\in\mathbb{Z}^+}} \frac{1}{y_1y_2\cdots y_k} &\le \frac{1}{1}\frac{(c_1\ln(m-1))^{k-2}}{m-1}+\cdots+\frac{1}{m-1}\frac{(c_1\ln 1)^{k-2}}{1}\\ &\le (c_1\ln m)^{k-2}\cdot\sum_{\substack{y_1+y_2=m \\ y_1,y_2\in\mathbb{Z}^+}}\frac{1}{y_1y_2}\\ &\le (c_1\ln m)^{k-2}\cdot\frac{c_1\ln m}{m}.\end{aligned}$$

Therefore the inequality holds for all k. $\square$

Lemma 2. *For any constant $\theta \in \mathbb{R}$ with $0 \le \theta < 1$, there exists a constant c_2 (may only depend on θ), such that for any constants $k, n' \in \mathbb{Z}^+$, $m \in \mathbb{R}$, and non-zero $\lambda_1, \lambda_2, \ldots, \lambda_k \in \mathbb{R}$, we have*

$$\sum_{\substack{\lambda_1y_1+\cdots+\lambda_ky_k=m \\ y_1,\ldots,y_k\in\{1,2,\ldots,n'\}}} \frac{1}{y_1^\theta y_2^\theta\cdots y_k^\theta} \le (c_2n')^{(k-1)(1-\theta)},$$

where the left side is considered to be 0 if there is no $y_1, y_2, \ldots, y_k \in \{1, 2, \ldots, n'\}$ satisfying $\lambda_1y_1+\lambda_2y_2+\cdots+\lambda_ky_k = m$.

Proof. For each tuple $(y_1, y_2, \ldots, y_{k-1}) \in \{1, 2, \ldots, n'\}^{k-1}$, there is at most one $y_k \in \{1, 2, \ldots, n'\}$ satisfying $\lambda_1y_1 + \lambda_2y_2 + \cdots + \lambda_ky_k = m$. Since $\frac{1}{y_k^\theta} \le 1$ because $0 \le \theta < 1$, we have

$$\begin{aligned}\sum_{\substack{\lambda_1y_1+\cdots+\lambda_ky_k=m \\ y_1,\ldots,y_k\in\{1,2,\ldots,n'\}}} \frac{1}{y_1^\theta y_2^\theta\cdots y_k^\theta} &\le \sum_{y_1,\ldots,y_{k-1}\in\{1,2,\ldots,n'\}}\frac{1}{y_1^\theta y_2^\theta\cdots y_{k-1}^\theta}\\ &= \left(\sum_{i=1}^{n'}\frac{1}{i^\theta}\right)^{k-1} \le (c_2n')^{(k-1)(1-\theta)},\end{aligned}$$

where c_2 is roughly $(\frac{1}{1-\theta})^{\frac{1}{1-\theta}}$. Therefore the lemma is proved. $\square$

Classification of paths. In a d-dimensional random graph $KSW(n, d, \gamma)$, there are two kinds of edges: *grid edges*, which are edges on the grid, and *long-range edges*, which are randomly added.

Fix two vertices u and v, a path from u to v may contain some long-range edges and some grid edges. We divide the path into several *segments* along the way from u to v: (a) each segment is either one long-range edge (called a *long-range segment*) or a batch of consecutive grid edges (called a *grid segment*); and (b) two consecutive segments cannot be both grid segments (otherwise combining them into one segment).

We use a d-dimensional vector to denote each edge, so that the source coordinate plus this vector equals to the destination coordinate module n'. For grid with wrap-around, there may be multiple vectors corresponding to one edge. We choose the vector in which every element is from $\{-\lfloor \frac{n'}{2} \rfloor, -\lfloor \frac{n'}{2} \rfloor + 1, \ldots, \lfloor \frac{n'-1}{2} \rfloor\}$. In this way, the vector representation of each edge is unique and the absolute value of every dimension is the smallest. We call this the *edge vector* of that edge. For every segment in the path, we call the summation (not module n') of all edge vectors the *segment vector*. For a vector $(x_1, x_2, \ldots, x_d)$, define its *sign pattern* as $(\mathrm{sgn}(x_1), \mathrm{sgn}(x_2), \ldots, \mathrm{sgn}(x_d))$.

We say two paths from u to v belong to the same *category* if (a) they have the same number of segments; (b) their corresponding segments are of the same type (long-range or grid segments); (c) for every pair of corresponding long-range segments in the two paths, the sign patterns of their segment vectors are the same; (d) for every pair of corresponding grid segments in the two paths, their segment vectors are equal; and (e) the summations (not module n') of all segment vectors in the two paths are equal.

The last condition is only used to distinguish paths that go different rounds in each dimension on grid with wrap-around.

In one category, there exist paths of which the long-range edges are identical but the grid edges may be different. To compute the probability of existing a path in a category, we only need to consider one path among the paths with identical long-range edges, since grid edges do not change probabilistic events and thus one such path exists if any only if other such paths exist.

We also assume that there are no repeated long-range edges in every path. For a path that has repeated long-range edges, we can obtain a shorter subpath without any repeated long-range edges so that the original path exists if and only if the new one exists. Since we are going to calculate the probability about paths not exceeding some length, it is safe to only consider paths without repeated long-range edges.

Lemma 3. *There exists a constant c_3 (dependent on d) such that for any fixed ℓ, the number of categories of paths from u to v of length ℓ is at most ${c_3}^{\ell}$.*

Proof. For each edge on the path, if it is a grid edge, it could be in one of the d dimensions, and in each dimension it could be in one of the two opposite directions, and thus a grid edge has $2d$ possibilities. If the edge is a long-range edge, on each dimension its sign has three possibilities $(+1, 0, -1)$, so totally 3^d possibilities for the sign pattern of the long-range segment vector. Moreover, in the grid with wrap-around, we consider each wrap-around of the path on some dimension to be one round in that dimension. Then the path can go at most $2\ell + 1$ different numbers of rounds on each dimension (ranging from ℓ rounds in one direction up to ℓ rounds in the other direction), so the summation of all segment vectors has at most $(2\ell+1)^d$ different values. The choice of each edge out of $2d + 3^d$ possibilities and the total summation of segments vectors determine a category. Therefore, the number of categories is bounded by $(2d + 3^d)^{\ell}(2\ell + 1)^d < {c_3}^{\ell}$ for some c_3. □

The above bound on the number of categories is not tight enough to be used for later analysis, when the number of long-range segments are small. Thus, we further bound the number of categories in the following way.

Lemma 4. *There exists a constant c_4 (dependent on d), such that for any fixed $\ell < n'$ and k with $1 \leq k \leq \ell$, the number of categories of paths from u to v of length ℓ and having k long-range segments is at most ${c_4}^k \ell^{(k+1)(d+1)}/k^{kd}$.*

Proof. For a path from u to v of length ℓ and containing k long-range segments ($1 \leq k \leq \ell < n'$), the summation of all segment vectors has at most $(2k+1)^d$ choices. This is because the path can go at most $2k+1$ different number of rounds on each dimension (k rounds in one direction to k rounds in the other direction). We consider the number of categories for a fixed summation of segment vectors first.

Suppose there are t grid segments, each having $a_1, a_2, \ldots, a_t$ edges respectively ($t \leq k+1$, $a_i \geq 1$, $a_1 + a_2 + \cdots + a_t < \ell$). If $t = 0$, then $k = \ell$, and it is easy to see that there are at most $(3^d)^k$ categories. Suppose now $t \geq 1$. For the i-th grid segment with a_i grid edges, its segment vector is such that on each dimension the only possible values are $-a_i, -a_i+1, \ldots, 0, \ldots, a_i - 1, a_i$. Thus, the number of possible segment vectors is $(2a_i+1)^d \leq 3^d a_i^d$. Since each long-range edge has 3^d possible sign patterns, the number of categories for fixed t and $a_1, a_2, \ldots, a_t$ is at most $(3^d)^k \prod_{i=1}^t 3^d a_i^d < 9^{(k+1)d}(\ell/t)^{td}$, where the inequality of arithmetic and geometry means is used.

The tuple $(a_1, a_2, \ldots, a_t)$ has less than ℓ^t possibilities. Considering $(2k+1)^d$ different possibilities of segment vectors summations, the total number of categories from u to v with length ℓ and containing k long-range edges is at most

$$\begin{aligned}
& (2k+1)^d \left\{ (3^d)^k + \sum_{t=1}^{k+1} \ell^t \cdot 9^{(k+1)d} (\ell/t)^{td} \right\} \\
< \;& (2k+1)^d \left\{ 3^{dk} + (k+1) 9^{(k+1)d} \max_{1 \leq t \leq k+1} \{\ell^t (\ell/t)^{td}\} \right\} \\
= \;& (2k+1)^d \left\{ 3^{dk} + (k+1) 9^{(k+1)d} \ell^{k+1} (\frac{\ell}{k+1})^{(k+1)d} \right\} \\
< \;& c_4^k \cdot \frac{\ell^{(k+1)(d+1)}}{k^{kd}},
\end{aligned}$$

where c_4 is a constant depending only on d. □

Probability calculation. We first give a lemma to calculate the probability of the existence of a specific edge. For an integer x, we define $\overline{x}$ to be $|x|$ if $x \neq 0$ and 1 if $x = 0$. We also define function $f(n')$ as follows:

$$f(n') = \begin{cases} \ln n' & \gamma = d, \\ (n')^{d-\gamma} & 0 \leq \gamma < d. \end{cases}$$

Lemma 5. *For two vertices u and v, the probability of the existence of a long-range undirected edge between u and v is at most $c_5(\overline{x_1} \cdot \overline{x_2} \cdots \overline{x_d})^{-\frac{\gamma}{d}}/f(n')$, where c_5 is a constant depending only on d and γ, and $(x_1, x_2, \ldots, x_d)$ is the edge vector if there exists a long-range edge from u to v and depends only on u and v.*

Proof. Say the non-zero elements of $(x_1, x_2, \ldots, x_d)$ are $(x_{i_1}, x_{i_2}, \ldots, x_{i_{d'}})$ ($d' \leq d$). Let p be the proba-

bility to add an edge from u to v. Then

$$p = \frac{(|x_1| + |x_2| + \cdots + |x_d|)^{-\gamma}}{\Theta(\sum_{i=1}^{n'} \frac{i^{d-1}}{i^{\gamma}})} = O\left(\frac{|x_{i_1} \cdot x_{i_2} \cdots x_{i_{d'}}|^{-\frac{\gamma}{d'}}}{\sum_{i=1}^{n'} i^{d-1-\gamma}}\right)$$
$$\leq O\left(\frac{|x_{i_1} \cdot x_{i_2} \cdots x_{i_{d'}}|^{-\frac{\gamma}{d}}}{\sum_{i=1}^{n'} i^{d-1-\gamma}}\right) = O\left(\frac{(\overline{x_1} \cdot \overline{x_2} \cdots \overline{x_d})^{-\frac{\gamma}{d}}}{\sum_{i=1}^{n'} i^{d-1-\gamma}}\right) = O\left(\frac{(\overline{x_1} \cdot \overline{x_2} \cdots \overline{x_d})^{-\frac{\gamma}{d}}}{f(n')}\right).$$

Moreover, the edge may also be from v to u, which also has probability p. By union bound the probability of the undirected edge (u, v) is $O(p)$. □

The following lemma gives the probability that one edge jumps within a local area.

Lemma 6. *For a vertex u and a long-range edge (u, v) from u, the probability that the grid distance between u, v is less than s is at most $c_6 f(s)/f(n')$, where c_6 is a constant depending only on d and γ.*

Proof. By union bound, the probability is at most

$$O\left(\sum_{i=1}^{s} i^{d-1} \frac{i^{-\gamma}}{f(n')}\right) = O\left(\sum_{i=1}^{s} \frac{i^{d-1-\gamma}}{f(n')}\right) \leq c_6 \frac{f(s)}{f(n')},$$

where c_6 is a constant. □

Given a path category $\mathcal{C}$, we now calculate the probability of existing a path in $\mathcal{C}$.

Lemma 7. *Given a path category $\mathcal{C}$ with length ℓ and k long-range edges. The probability that there exists a path in $\mathcal{C}$ is at most*

$$\begin{cases} c_5^k (c_7^k k^k)^d \Big/ (\ln n')^k & \gamma = d, \\ c_5^k c_2^{(k-1)(d-\gamma)} \Big/ (n')^{d-\gamma} & 0 \leq \gamma < d, \end{cases}$$

where c_2 and c_5 are the constants given in Lemma 5 and c_7 is another constant.

Proof. Let the segment vectors of the long-range edges in a path $P \in \mathcal{C}$ be

$$(x_{11}, x_{12}, \ldots, x_{1d}), (x_{21}, x_{22}, \ldots, x_{2d}), \ldots, (x_{k1}, x_{k2}, \ldots, x_{kd}).$$

By our definition of a category, all paths in $\mathcal{C}$ have the same sign patterns on the corresponding long-range segment vectors. Thus, we can define the following sets for the category $\mathcal{C}$: $A_i^+ = \{j \mid x_{ji} > 0, 1 \leq j \leq k\}$, $A_i^- = \{j \mid x_{ji} < 0, 1 \leq j \leq k\}$, and $A_i^0 = \{j \mid x_{ji} = 0, 1 \leq j \leq k\}$, for all $1 \leq i \leq d$. For fixed u and v, there is a fixed vector $(t_1, t_2, \ldots, t_d)$ such that the summation of the segment vectors of all long-range segments in any $P \in \mathcal{C}$ is vector $(t_1, t_2, \ldots, t_d)$. This is because the summation of all segment vectors from u to v is fixed, and all grid segments have the fixed segment vectors. Therefore, a path in $\mathcal{C}$ can be characterized by kd integers $x_{11}, \ldots, x_{kd}$ satisfying the following for all $1 \leq i \leq d$:

$$\begin{cases} x_{ji} \in \{1, \ldots, n'\} & \text{for } j \in A_i^+, \\ x_{ji} \in \{-n', \ldots, -1\} & \text{for } j \in A_i^-, \\ x_{ji} = 0 & \text{for } j \in A_i^0, \\ \displaystyle\sum_{j \in A_i^+} |x_{ji}| - \sum_{j \in A_i^-} |x_{ji}| = t_i. \end{cases} \tag{1}$$

The probability that some path exists is the multiplication of the probability of the first edge, the probability of the second edge conditioned on the existence of the first edge, the probability of the third edge conditioned on the existence of the first two edge, etc. In our model, the probability of an (undirected) edge conditioned on the existence of other (undirected) edges is less than or equal to the probability without condition, because each vertex can only connect to exact one other vertex (when considering the edge direction). Hence we can use the multiplication of the probabilities of all edges as an upper bound of the probability of a path. By union bound and Lemma 5, the probability that a path exists in $\mathcal{C}$ is at most

$$\begin{aligned}
& \sum_{\text{all paths in } \mathcal{C}} \prod_{j=1}^{k} \frac{c_5(\overline{x_{j1}} \cdot \overline{x_{j2}} \cdots \overline{x_{jd}})^{-\frac{\gamma}{d}}}{f(n')} \\
\leq \; & \frac{c_5^k}{f^k(n')} \sum_{\substack{x_{11},\ldots,x_{kd}: \\ \text{satisfying (1)}}} \prod_{j=1}^{k} (\overline{x_{j1}} \cdot \overline{x_{j2}} \cdots \overline{x_{jd}})^{-\frac{\gamma}{d}} \\
= \; & \frac{c_5^k}{f^k(n')} \cdot \prod_{i=1}^{d} \left\{ \sum_{\substack{\overline{x_{1i}},\ldots,\overline{x_{ki}}: \sum_{j\in A_i^+} \overline{x_{ji}} - \sum_{j\in A_i^-} \overline{x_{ji}} = t_i; \\ \text{for } j\in A_i^0,\ \overline{x_{ji}}=1; \text{ for } j\in A_i^+\cup A_i^-,\ \overline{x_{ji}}\in\{1,\ldots,n'\}}} \left(\prod_{j=1}^{k} \overline{x_{ji}}\right)^{-\frac{\gamma}{d}} \right\} \\
= \; & \frac{c_5^k}{f^k(n')} \cdot \prod_{i=1}^{d} \left\{ \sum_{\substack{\overline{x_{ji}} \ (j \text{ ranges in } A_i^+\cup A_i^-): \\ \sum_{j\in A_i^+} \overline{x_{ji}} - \sum_{j\in A_i^-} \overline{x_{ji}} = t_i; \text{ and } \overline{x_{ji}}\in\{1,\ldots,n'\}}} \left(\prod_{j\in A_i^+\cup A_i^-} \overline{x_{ji}}\right)^{-\frac{\gamma}{d}} \right\}.
\end{aligned} \tag{2}$$

The inner brace summation is considered to be 1 for $A_i^+ = A_i^- = \emptyset$. Now we consider the following sum for disjoint sets $A^+, A^- \subseteq \{1, 2, \ldots, k\}$ (at least one is not empty) and numbers $t \in \mathbb{Z}$, $\gamma \geq 0$, $d \geq 0$,

$$\sum_{\substack{y_j \ (j \text{ ranges in } A^+\cup A^-): \\ \sum_{j\in A^+} y_j - \sum_{j\in A^-} y_j = t; \text{ and } y_j\in\{1,2,\ldots,n'\}}} \left(\prod_{j\in A^+\cup A^-} y_j\right)^{-\frac{\gamma}{d}}. \tag{3}$$

- Case $\gamma = d$.

  If one of A^+ and A^- is $\emptyset$, we only need to consider the case that $t \neq 0$, because the sum (3) is 0 for $t = 0$. By Lemma 1, the sum (3) is bounded by

  $$c_1^k \frac{(\ln |t|)^k}{|t|} \leq c_1^k \max_{x\geq 1}\{\frac{(\ln x)^k}{x}\} = (c_1 k/e)^k.$$

  If neither A^+ nor A^- is $\emptyset$, we assume that $t \geq 0$. (The calculation for $t < 0$ is similar if we exchange

A^+ and A^-.) By Lemma 1 the sum (3) is bounded by

$$\begin{aligned}
& \sum_{s=1}^{kn'} \left(\sum_{\sum_{j\in A^-} y_j = s} \frac{1}{\prod_{j\in A^-} y_j} \right) \left(\sum_{\sum_{j\in A^+} y_j = s+t} \frac{1}{\prod_{j\in A^+} y_j} \right) \\
\leq & \sum_{s=1}^{kn'} \frac{(c_1 \ln s)^{|A^-|-1}}{s} \cdot \frac{(c_1 \ln(s+t))^{|A^+|-1}}{s+t} \\
\leq & \; c_1^k \sum_{s=1}^{kn'} \frac{(\ln(s+t))^k}{s(s+t)} \\
= & \; c_1^k \sum_{s=1}^{kn'} \left(\frac{(\ln(s+t))^k}{\sqrt{s+t}} \cdot \frac{1}{s\sqrt{s+t}} \right) \\
\leq & \; c_1^k \cdot \max_{x\geq 1} \{ \frac{(\ln x)^k}{x^{0.5}} \} \cdot \sum_{s=1}^{\infty} \frac{1}{s^{1.5}} \\
= & \; c_1^k \cdot \frac{(2k)^k}{e^k} \cdot O(1).
\end{aligned}$$

In either case, the sum (3) is bounded by $c_7^k k^k$ for some constant c_7. By (2) the probability of a path in $\mathcal{C}$ is at most $c_5^k (c_7^k k^k)^d / (\ln n')^k$.

- Case $0 \leq \gamma < d$. By Lemma 2, the sum (3) is bounded by

  $$(c_2 n')^{(|A^+|+|A^-|-1)(1-\frac{\gamma}{d})} \leq (c_2 n')^{(k-1)(1-\frac{\gamma}{d})}.$$

  Therefore, by (2) the probability of a path in $\mathcal{C}$ is at most

  $$\frac{c_5^k}{(n')^{k(d-\gamma)}} \cdot \left((c_2 n')^{(k-1)(1-\frac{\gamma}{d})} \right)^d = \frac{c_5^k c_2^{(k-1)(d-\gamma)}}{(n')^{d-\gamma}}.$$

$\square$

Finally, we apply Lemmas 3, 4, 6, and 7 together to show that the probability of any two vertices connected by a short path with at least one long-range edge is diminishingly small.

Lemma 8. *For any two vertices u and v, the probability that a path exists connecting u and v with at least one long-range edge and total length at most ℓ is $o(1)$, when $\ell \leq (\log n')^{\frac{1}{1.5(d+1)+\varepsilon}}$ ($\varepsilon > 0$) for $\gamma = d$; and $\ell < c \log n'$ (c is some constant depending only on d and γ) for $0 \leq \gamma < d$.*

Proof. We first study the probability of a path with exact length ℓ, and we divide it into the following two cases.

- Case $\gamma = d$. If $k = 1$, the probability that a path from u to v with length ℓ exists is at most $O(\ell^d c_6 \ln(3\ell)/\ln n') = O(\ell^d \ln \ell / \ln n')$. To see this, we divide the path into three segments: a grid segment followed by one long-range edge, then followed by another grid segments. The first grid

segment can reach at most $O(\ell^d)$ destinations. For each of this destination w, the long-range edge has to reach some vertex within grid distance ℓ of vertex v. And by triangle inequality, it must be within grid distance 3ℓ of vertex w since the distances between v, u and u, w are both at most ℓ. By Lemma 6, we know this probability is $c_6 \ln(3\ell)/\ln n'$. Therefore, the above statement holds.

For $k \geq 2$, we simply combine Lemmas 4 and 7. Thus, the probability that a path with length ℓ exists is at most

$$\begin{aligned} & O\left(\frac{\ell^d \ln \ell}{\ln n'}\right) + \sum_{k=2}^{\ell} \frac{c_4^k \ell^{(k+1)(d+1)}}{k^{kd}} \cdot \frac{c_5^k \left(c_7^k k^k\right)^d}{(\ln n')^k} \\ = \quad & O\left(\frac{\ell^d \ln \ell}{\ln n'}\right) + \sum_{k=2}^{\ell} \left(\frac{c_4 c_5 c_7^d \cdot \ell^{(d+1)(1+\frac{1}{k})}}{\ln n'}\right)^k \\ \leq \quad & O\left(\frac{\ell^d \ln \ell}{\ln n'}\right) + \sum_{k=2}^{\ell} \left(\frac{c_4 c_5 c_7^d \cdot \ell^{1.5(d+1)}}{\ln n'}\right)^k . \end{aligned}$$

This probability is $o(1)$ when $\ell \leq (\log n')^{\frac{1}{1.5(d+1)+\varepsilon}}$ for any $\varepsilon > 0$.

- Case $0 \leq \gamma < d$. In this case, we combine Lemmas 3 and 7. The probability that a path with length ℓ exists is at most

$$c_3{}^{\ell} \cdot \max_{1 \leq k \leq \ell} \left\{ \frac{c_5^k c_2^{(k-1)(d-\gamma)}}{(n')^{d-\gamma}} \right\} \leq \frac{(c_3 c_5 c_2^{d-\gamma})^{\ell}}{(n')^{d-\gamma}},$$

  where the inequality is based on $c_2, c_5 \geq 1$, which is obviously the case. This probability is $o(1)$ when $\ell < c \log n'$ for some properly chosen constant c, which depends only on d and γ.

We now consider the case of path length less than ℓ. Let w be a grid neighbor of v. For any path connecting u and v with length less than ℓ, we can add grid edges (v, w) followed by (w, v), to increase the path length to either $\ell - 1$ or ℓ. Thus, the probability that a path of length at most ℓ exists is the same as the probability that a path of length $\ell - 1$ or ℓ exists. The case of $\ell - 1$ follows exactly the same as the case of ℓ argued above. Therefore, the lemma holds. □

With Lemma 8, we are ready to prove Theorem 1 for the case of $d \geq 2$ and $0 \leq \gamma \leq d$.

Proof of Theorem 1 for the case of $d \geq 2$ and $0 \leq \gamma \leq d$. We define $\ell = \lfloor (\log n')^{\frac{1}{1.5(d+1)+\varepsilon}}/2 \rfloor$ when $\gamma = d$, and $\ell = \lfloor c \log n'/2 \rfloor$ when $0 \leq \gamma < d$, where c is the constant determined by Lemma 8. In the base grid, find any square in any dimension with one side length equal to ℓ. Consider the four corner vertices of the square. By Lemma 8, the probability that a pair of corners of this square are connected by a path with at least one long-range edge and total length at most 2ℓ is $o(1)$. Thus by union bound, the distances between any pair of these four corner vertices are exactly their grid distance, with probability $1 - o(1)$. This means that the δ value of these four vertices is ℓ, with probability $1 - o(1)$. Therefore, we know that with probability $1 - o(1)$, $\delta(KSW(n, d, \gamma)) = \Omega((\log n')^{\frac{1}{1.5(d+1)+\varepsilon}}) = \Omega((\log n)^{\frac{1}{1.5(d+1)+\varepsilon}})$ when $d \geq 2$ and $\gamma = d$, and $\delta(KSW(n, d, \gamma)) = \Omega(\log n') = \Omega(\log n)$ when $d \geq 2$ and $0 \leq \gamma < d$. □

Remark (The limitation of this approach). We already have tight lower bound for $0 \leq \gamma < d$. However, the lower bound $\Omega((\log n)^{\frac{1}{1.5(d+1)+\varepsilon}})$ for $\gamma = d$ is still not matching the upper bound $O(\log n)$. We show

below that the lower bound can never be improved to $\Omega(\log n)$ by the above technique of proving that the grid distance is the shortest on every small square (with high probability).

Consider a *KSW* graph with $\gamma = d$. For an $\ell \times \ell$ square S, let u be its the upper left vertex and v be its lower right vertex. Similar to the proof in Lemma 6, the probability of existence of an edge linking any w into the $\ell/2 \times \ell/2$ square on the lower right side of w within the square S (but excluding w's two grid neighbors in the square) is

$$q = \Theta\left(\sum_{i=2}^{\ell/2} i \cdot \frac{i^{-d}}{\ln n'}\right) = \begin{cases} \Theta(\frac{\log \ell}{\log n'}) & d = 2, \\ \Theta(\frac{1}{\log n'}) & d \geq 3. \end{cases}$$

Consider the $\ell/2 \times \ell/2$ square on the lower right side of u, which is the upper left quadrant of the original square. The probability that at least one vertex w in this quadrant links to its lower right $\ell/2 \times \ell/2$ square is $1-(1-q)^{\ell/2 \times \ell/2}$. This probability is almost 1 when $\ell = \omega\left(\sqrt{\frac{\log n'}{\log\log n'}}\right)$ for $d = 2$, or $\ell = \omega(\sqrt{\log n'})$ for $d \geq 3$. If there exists such a vertex w in the upper right quadrant, and suppose it links to a vertex x in its lower right $\ell/2 \times \ell/2$ square. Then x must also be in the original square S, and we can have a path from u to w following the grid path, then the long-range edge from w to x, and then the grid path from x to v. This path connects u and v and must be shorter than the grid paths from u to v. Therefore our technique cannot improve the lower bound to $\omega\left(\sqrt{\frac{\log n'}{\log\log n'}}\right) = \omega\left(\sqrt{\frac{\log n}{\log\log n}}\right)$ for $d = 2$ or $\omega(\sqrt{\log n'}) = \omega(\sqrt{\log n})$ for $d \geq 3$.

3.3.2 The case of $d = 1$ and $0 \leq \gamma \leq 1$

In this section we give lower bounds of δ for the one dimensional *KSW* model (based on an n-vertex ring). Let the n vertices be $v_0, \ldots, v_{n-1}$. Let $\ell_0 = \lfloor (\log n)^{\frac{1}{1.5(d+1)+\varepsilon}} \rfloor$ when $\gamma = 1$, or $\lfloor c \log n \rfloor$ when $0 \leq \gamma < 1$, where c is the number determined in Lemma 8.

The idea is to find a long-range edge e_0 between two vertices with grid distance ℓ_0. As e_0 forms a local ring with the original grid, we can give a lower bound of δ such that the ring distances (with respect to the grid edges and e_0) are the shortest even after adding the long-range edges. We first calculate the probability of ring distances being the shortest under the condition of existing e_0.

We divide the construction of a *KSW* graph into two stages: 1) Every vertex links to exactly one other vertex according to some distribution; and 2) we ignore the edge direction and consider the graph as undirected. Let $\mathcal{E}_i$ ($0 \leq i \leq n-1$) be the event that v_i links to $v_{(i+\ell_0 \bmod n)}$ in the first stage. Under the condition that $\mathcal{E}_i$ happens, $v_{(i+\ell_0 \bmod n)}$ is still free to link to any vertex. The events $\mathcal{E}_0, \mathcal{E}_1, \ldots, \mathcal{E}_{n-1}$ are independent.

Lemma 9. *Under the condition that $\mathcal{E}_i$ happens (v_i links to $v_{(i+\ell_0 \bmod n)}$), for any two vertices u, w on the curve from v_i to $v_{(i+\ell_0 \bmod n)}$ (exclusive), the conditional probability of existing a path connecting u and w with at least one long-range edge other than $(v_i, v_{(i+\ell_0 \bmod n)})$ and path length at most ℓ_0 is $o(1)$.*

Proof. Let e_0 be the edge $(v_i, v_{(i+\ell_0 \bmod n)})$. We still follow the arguments for $d \geq 2$ and $0 \leq \gamma \leq d$, but change the classification of paths slightly: now e_0 is a type of edges by itself, and thus together with grid edges and long-range edges, we have three types of edges and three types of corresponding segments. For each original category of paths with length ℓ and k long-range segments defined in the previous section, we further divide it into at most $k+1$ categories, based on whether e_0 was the first, second, ..., or the k-th long-range segment, or e_0 does not appear in the path. The number of long-range edges in a category with e_0 is decreased by 1. For the categories with only grid edges and e_0, those paths now have no long-range edge and they exist with conditional probability 1, hence they will not be calculated.

For Lemma 3, the number of categories is increased by a factor of at most $\ell + 1$. For Lemma 4, the number of categories is increased by a factor of at most $k + 1$. Thus, we only need to properly adjust the constants c_3 and c_4 in these two lemmas to make them still hold. Lemma 5 is not changed except that e_0 is no longer a long-range edge and the lemma is not applicable on e_0. Lemma 6 is not affected. In the proof of Lemma 7, a category without e_0 can be calculated normally. For a category with e_0, the number of long-range edges is decreased by 1 as stated above. One can see that all the arguments still hold by changing the summation of long-range edges (t_i in (1)) to contain e_0.

For Lemma 8, only the case $\gamma = d$ and $k = 1$ (one long-range edge) needs some modification. If the path does not contain e_0, the calculation still holds. Otherwise, we can divide the path into five segments: grid, long-range, grid, e_0, grid (or grid, e_0, grid, long-range, grid). We just consider the 3 consecutive segments grid, e_0, grid as a whole, which contains $O(\ell)$ edges and can reach $O(\ell^d) = O(\ell)$ destinations. One can see the previous argument still holds. Therefore, the consequence of Lemma 8 still holds. □

With Lemma 9, we give the proof of Theorem 1 for the case of $d = 1$ and $0 \le \gamma \le 1$.

Proof of Theorem 1 for the case of $d = 1$ *and* $0 \le \gamma \le 1$. Let $\mathcal{F}$ be the event $\delta \ge \frac{1}{4}\ell_0 - 3$. We first show that $\Pr\{\mathcal{F} \mid \mathcal{E}_i\} = 1 - o(1)$. Pick four vertices $A_i = v_{(i+\lfloor \frac{1}{8}\ell_0 \rfloor \bmod n)}$, $B_i = v_{(i+\lfloor \frac{3}{8}\ell_0 \rfloor \bmod n)}$, $C_i = v_{(i+\lfloor \frac{5}{8}\ell_0 \rfloor \bmod n)}$ and $D_i = v_{(i+\lfloor \frac{7}{8}\ell_0 \rfloor \bmod n)}$. By Lemma 9 and union bound, we know that with probability $1 - o(1)$, the distances between every pair of vertices are the ring (grid edges plus $(v_i, v_{(i+\ell_0 \bmod n)})$) distances. That is, the distances between A_iB_i, B_iC_i, C_iD_i, D_iA_i are roughly $\frac{1}{4}\ell_0$ (off by at most 2), and the distances between A_iC_i and B_iD_i are roughly $\frac{1}{2}\ell_0$ (off by at most 1). Therefore, by considering the four vertices $A_iB_iC_iD_i$, we have δ at least $\frac{1}{2}(\frac{1}{2}\ell_0 - 1 + \frac{1}{2}\ell_0 - 1 - \frac{1}{4}\ell_0 - 2 - \frac{1}{4}\ell_0 - 2) = \frac{1}{4}\ell_0 - 3$ with conditional probability $1 - o(1)$.

For every $\mathcal{E}_i$, the probability that v_i links to $v_{(i+\ell_0 \bmod n)}$ is

$$\Pr\{\mathcal{E}_i\} = \Theta(\frac{\ell_0^{-\gamma}}{f(n)}) = \begin{cases} \Theta((\log n)^{-\frac{2+\varepsilon}{3+\varepsilon}}) & \gamma = 1, \\ \Theta(\log n / n^{1-\gamma}) & 0 \le \gamma < 1. \end{cases}$$

Define this probability as q. We have $\Pr\{\mathcal{F} \text{ and } \mathcal{E}_i\} = \Pr\{\mathcal{F} \mid \mathcal{E}_i\} \Pr\{\mathcal{E}_i\} = (1 - o(1))q$.

Let K be the random variable denoting the number of $\mathcal{E}_i$'s that occur. We define $m = E[K]$, and we have $m = nq$. One can check that $m = poly(n)$ for both cases $\gamma = 1$ and $0 \le \gamma < 1$. By Chernoff Bound, $\Pr\{|K - m| \le m^{0.6}\} > 1 - 2e^{-(m^{-0.4})^2/4 \cdot m} = 1 - 2e^{-m^{0.2}/4}$. Hence with very high probability, K is close to m. Let $\mathcal{G}$ denote the event that $m - m^{0.6} \le K \le m + m^{0.6}$. We show that $\Pr\{\mathcal{F} \text{ and } \mathcal{E}_i \text{ and } \mathcal{G}\}$ is very close to $\Pr\{\mathcal{F} \text{ and } \mathcal{E}_i\}$. In fact, $\Pr\{\mathcal{F} \text{ and } \mathcal{E}_i \text{ and } \mathcal{G}\} \ge \Pr\{\mathcal{F} \text{ and } \mathcal{E}_i\} - \Pr\{\text{not } G\} = (1 - o(1)) \Pr\{\mathcal{F} \text{ and } \mathcal{E}_i\}$, because $\Pr\{\text{not } \mathcal{G}\} < 2e^{-m^{0.2}/4} = 2e^{-poly(n)}$ is much smaller than $\Pr\{\mathcal{F} \text{ and } \mathcal{E}_i\} = (1 - o(1))q = \omega(1/n)$. On the other hand, it is straightforward that $\Pr\{\mathcal{F} \text{ and } \mathcal{E}_i \text{ and } \mathcal{G}\} \le \Pr\{\mathcal{F} \text{ and } \mathcal{E}_i\}$. Hence $\Pr\{\mathcal{F} \text{ and } \mathcal{E}_i \text{ and } \mathcal{G}\}$ differs from $\Pr\{\mathcal{F} \text{ and } \mathcal{E}_i\}$ by a factor at most $(1 - o(1))$, and $\Pr\{\mathcal{F} \text{ and } \mathcal{E}_i \text{ and } \mathcal{G}\} = (1 - o(1))q$.

For every $r = r_0 r_1 \cdots r_{n-1} \in \{0,1\}^n$, define $\mathcal{H}_r$ to be the event that $\mathcal{E}_j$ happens iff $r_j = 1$ for all $j = 0, 1, \ldots, n-1$. We can see $\mathcal{H}_r$'s are mutually exclusive for $r \in \{0,1\}^n$. Hence

$$\Pr\{\mathcal{F} \text{ and } \mathcal{E}_i \text{ and } \mathcal{G}\} = \sum_{k=m-m^{0.6}}^{m+m^{0.6}} \sum_{\substack{r \in \{0,1\}^n \\ r_i = 1 \text{ and } r \text{ has } k \text{ 1's}}} \Pr\{\mathcal{F} \text{ and } \mathcal{H}_r\}.$$

Therefore

$$\begin{aligned}
\Pr\{\mathcal{F} \text{ and } \mathcal{G}\} &= \sum_{k=m-m^{0.6}}^{m+m^{0.6}} \sum_{\substack{r\in\{0,1\}^n \\ r \text{ has } k \text{ 1's}}} \Pr\{\mathcal{F} \text{ and } \mathcal{H}_r\} \\
&= \sum_{k=m-m^{0.6}}^{m+m^{0.6}} \frac{1}{k} \cdot \sum_{i=0}^{n-1} \sum_{\substack{r\in\{0,1\}^n \\ r_i=1 \text{ and } r \text{ has } k \text{ 1's}}} \Pr\{\mathcal{F} \text{ and } \mathcal{H}_r\} \\
&> \frac{1}{m+m^{0.6}} \cdot \sum_{i=0}^{n-1} \sum_{k=m-m^{0.6}}^{m+m^{0.6}} \sum_{\substack{r\in\{0,1\}^n \\ r_i=1 \text{ and } r \text{ has } k \text{ 1's}}} \Pr\{\mathcal{F} \text{ and } \mathcal{H}_r\} \\
&= \frac{1}{m+m^{0.6}} \sum_{i=0}^{n-1} \Pr\{\mathcal{F} \text{ and } \mathcal{E}_i \text{ and } \mathcal{G}\} \\
&= \frac{n(1-o(1))q}{m+m^{0.6}} = \frac{m(1-o(1))}{m+m^{0.6}} = 1-o(1).
\end{aligned}$$

The probability that $\delta \geq \frac{1}{4}\ell_0 - 3 = \Omega(\ell_0)$ is $\Pr\{\mathcal{F}\} \geq \Pr\{\mathcal{F} \text{ and } \mathcal{G}\} = 1 - o(1)$. ☐

3.3.3 The case of $d = 1$ and $\gamma > 3$

We first show that with high probability all long-range edges connect two vertices with grid distance $o(n)$, for general d and $\gamma > 2d$.

Lemma 10. *In a random graph from $KSW(n, d, \gamma)$ with $\gamma > 2d$, with probability $1 - o(1)$ there is no long-range edge that connects two vertices with grid distance larger than $n^{\frac{1}{\gamma-d}+\varepsilon}$, where ε is any positive number.*

Proof. For a vertex u, the probability that the long-range edge from u links to somewhere with distance longer than $\ell_0 = n^{\frac{1}{\gamma-d}+\varepsilon}$ from u is

$$O\left(\sum_{i=\ell_0}^{n^{1/d}} i^{d-1} \frac{i^{-\gamma}}{\sum_{j=1}^{n^{1/d}} j^{d-1} j^{-\gamma}}\right) = O\left(\sum_{i=\ell_0}^{\infty} i^{d-1-\gamma}\right) = O\left(\ell_0^{d-\gamma}\right).$$

By union bound, the probability that such u exists is

$$O\left(n\ell_0^{d-\gamma}\right) = O\left(n \cdot n^{(\frac{1}{\gamma-d}+\varepsilon)(d-\gamma)}\right) = O\left(n^{\varepsilon(d-\gamma)}\right) = o(1).$$

Therefore with probability $1 - o(1)$ such u does not exist. ☐

Given a graph G in $KSW(n, d, \gamma)$, let $\ell_0(G)$ be the largest grid distance of two vertices connected by a long-range edge in G. From the above result, we know that when $\gamma > 3d$, $\ell_0(G) < n^{\frac{1}{\gamma-d}+\varepsilon} = o(\sqrt{n^{1/d}})$ with high probability. For the rest of this section, with $d = 1$, we fix G to be any graph in $KSW(n, 1, \gamma)$ with $\ell_0(G) < n^{\frac{1}{\gamma-1}+\varepsilon}$, and show that $\delta(G) = \Omega(n^c)$ for some constant c. Since G is fixed, we will use ℓ_0 to be the short hand of $\ell_0(G)$.

Now go back to the one dimensional grid with wrap-around, which is a ring with vertices $v_0, v_1, \ldots, v_{n-1}$. Notice that in one-dimensional case, the edge vector defined in Section 3.3.1 degenerates to a scalar value from $\{-\lfloor \frac{n}{2} \rfloor, -\lfloor \frac{n}{2} \rfloor + 1, \ldots, \lfloor \frac{n-1}{2} \rfloor\}$. We arrange $v_0, v_1, \ldots, v_{n-1}$ clockwise on the ring. Then a positive edge scalar corresponds to a clockwise hop while a negative edge scalar corresponds to a counter-clockwise hop.

Let $A = v_0$ and $B = v_{\lfloor n/2 \rfloor}$ be two specific vertices. We define two kinds of paths between A and B: a *positive path* is one in which the summation of edge scalars (not taking module n) is positive, while a *negative path* is one in which the summation of edge scalars (not taking module n) is negative.

Lemma 11. *There exists a positive path from A to B that does not go through* $v_{\lfloor n/2 \rfloor+1}, v_{\lfloor n/2 \rfloor+2}, \ldots, v_{n-1}$, *and the length is at most* $2\ell_0$ *longer than the shortest positive path from A to B. Similarly, there exists a negative path from A to B that does not go through* $v_1, v_2, \ldots, v_{\lfloor n/2 \rfloor-1}$, *and the length is at most* $2\ell_0$ *longer than the shortest negative path from A to B.*

Proof. We only give the proof for the positive path case. Consider any shortest positive path $\mathcal{P}$ from A to B. We first show the following claim. Let $S_A = \{v_1, \ldots, v_{\ell_0}\}$ be the set of ℓ_0 consecutive vertices clockwise to A, and $S_B = \{v_{\lfloor n/2 \rfloor - \ell_0}, v_{\lfloor n/2 \rfloor - \ell_0 + 1}, \ldots, v_{\lfloor n/2 \rfloor - 1}\}$ be the set of ℓ_0 consecutive vertices counter-clockwise to B.

Claim. There must exist a subpath in $\mathcal{P}$ from a vertex $u \in S_A$ to a vertex $w \in S_B$ that does not go through $v_{\lfloor n/2 \rfloor+1}, v_{\lfloor n/2 \rfloor+2}, \ldots, v_{n-1}$.

Say there are m edges in $\mathcal{P}$. Let s_i ($0 \leq i \leq m$) denote the summation of the first i edge scalars in $\mathcal{P}$ (not taking module n). Initially we have $s_0 = 0$, and the final value s_m is $kn + \lfloor n/2 \rfloor$ for some integer $k \geq 0$. One can also see that the position after going through the first i edges in $\mathcal{P}$ is at $v_{(s_i \bmod n)}$. Let i_1 be the smallest integer such that $s_{i_1} \geq \lfloor n/2 \rfloor$ (exist because $s_m \geq \lfloor n/2 \rfloor$), and i_2 be the largest integer such that $i_2 < i_1$ and $s_{i_2} \leq 0$ (exist because $s_0 = 0$).

We consider the (i_2+1)-th edge, which begins at $v_{(s_{i_2} \bmod n)}$ for some $s_{i_2} \leq 0$, and ends at $v_{(s_{i_2+1} \bmod n)}$ for some $s_{i_2+1} > 0$. The number s_{i_2+1} is at most $s_{i_2} + \ell_0$ since no edge is longer than ℓ_0. Therefore, s_{i_2+1} must be a number in $(0, \ell_0]$ and $v_{(s_{i_2+1} \bmod n)}$ must be in S_A. We choose $u = v_{(s_{i_2+1} \bmod n)} \in S_A$. Similarly, pick $w = s_{i_1-1}$, which is the beginning point of the i_1-th edge, we have $w \in S_B$. The intermediate values $s_{i_2+1}, s_{i_2+2}, \ldots, s_{i_1-1}$ are all in the interval $(0, \lfloor n/2 \rfloor)$ by the definitions of i_1 and i_2. That is, for all j such that $i_2 + 1 \leq j \leq i_1 - 1$, $s_j \bmod n = s_j$, and the corresponding vertex $v_{(s_j \bmod n)} = v_{s_j} \in \{v_1, v_2, \ldots, v_{\lfloor n/2 \rfloor - 1}\}$. Therefore the subpath from u to w does not go through $v_{\lfloor n/2 \rfloor+1}, v_{\lfloor n/2 \rfloor+2}, \ldots, v_{n-1}$, and the claim holds.

With the claim, we can construct a positive path $\mathcal{P}'$, which use ring edges from A to u, then use the subpath in the claim from u to w, and then from w to B using ring edges. The length of $\mathcal{P}'$ is at most $2\ell_0$ longer than $\mathcal{P}$, the shortest positive path from A to B. ☐

We use $\mathcal{P}^+_{AB}$ and $\mathcal{P}^-_{AB}$ to denote the two paths stated in the above lemma. According to this lemma, one of $\mathcal{P}^+_{AB}$ and $\mathcal{P}^-_{AB}$ is at most $2\ell_0$ longer than the shortest path between A and B.

Let C be the middle point of $\mathcal{P}^+_{AB}$ (take a vertex nearest middle if the path has odd number of edges), and $\mathcal{P}^+_{AC}$, $\mathcal{P}^+_{CB}$ be the two subpaths from A to C and C to B. Similarly, let D be the middle point of $\mathcal{P}^-_{AB}$ and $\mathcal{P}^-_{AD}$, $\mathcal{P}^-_{DB}$ be the two subpaths.

Lemma 12. *The paths* $\mathcal{P}^+_{AC}$, $\mathcal{P}^+_{CB}$, $\mathcal{P}^-_{AD}$ *and* $\mathcal{P}^-_{DB}$ *are at most* $3\ell_0 + 1$ *longer than the shortest paths between corresponding pairs of vertices.*

Proof. We only give the proof for $\mathcal{P}^+_{AC}$. Suppose that it is not true, and the shortest path $\mathcal{P}^*_{AC}$ from A to C is at least $3l_0 + 2$ shorter than $\mathcal{P}^+_{AC}$. The path $\mathcal{P}^*_{AC}$ must be at least $3\ell_0 + 1$ shorter than $\mathcal{P}^+_{CB}$ because C is the point nearest middle of $\mathcal{P}^+_{AB}$.

Consider the last time that the path $\mathcal{P}^*_{AC}$ gets into the range $\{v_0, v_1, \ldots, v_{\lfloor n/2 \rfloor}\}$, the subpath of $\mathcal{P}^*_{AC}$ from that point to C must be one of the following cases.

- It is a subpath from a vertex $A' \in \{v_0, v_1, \ldots, v_{\ell_0}\}$ to C not going through $v_{\lfloor n/2 \rfloor+1}$, $v_{\lfloor n/2 \rfloor+2}$, $\ldots$,v_{n-1}. Replace $\mathcal{P}^+_{AC}$ by the path from A to A' through ring edges concatenated with the subpath of $\mathcal{P}^*_{AC}$ from A' to C. This will cause the length of $\mathcal{P}^+_{AB}$ to decrease by at least $3\ell_0 + 2 - \ell_0 > 2\ell_0$, which is impossible by Lemma 11.

- It is a subpath from a vertex $B' \in \{v_{\lfloor n/2 \rfloor - \ell_0}, v_{\lfloor n/2 \rfloor - \ell_0 + 1}, \ldots, v_{\lfloor n/2 \rfloor}\}$ to C that does not go through $v_{\lfloor n/2 \rfloor+1}$, $v_{\lfloor n/2 \rfloor+2}$, $\ldots$,v_{n-1}. Replace $\mathcal{P}^+_{CB}$ by the reverse of this subpath of $\mathcal{P}^*_{AC}$ from C to B' concatenated with ring edges from B' to B. This will cause the length of $\mathcal{P}^+_{AB}$ to decrease by at least $3\ell_0 + 1 - \ell_0 > 2\ell_0$, which is impossible by Lemma 11.

Therefore the lemma holds. $\square$

Then we consider the shortest path between C and D.

Lemma 13. *Either the concatenation of $\mathcal{P}^+_{CB}$ and reversed $\mathcal{P}^-_{DB}$, or the concatenation of reversed $\mathcal{P}^+_{AC}$ and $\mathcal{P}^-_{DA}$ is at most $8\ell_0 + 2$ longer than the shortest path between C and D.*

Proof. The shortest path from C to D (say $\mathcal{P}^*_{CD}$) must go through either B's neighborhood $v_{\lfloor n/2 \rfloor}$, $v_{\lfloor n/2 \rfloor+1}$, $\ldots$, $v_{\lfloor n/2 \rfloor+\ell_0}$ or A's neighborhood $v_0, v_1, \ldots, v_{\ell_0}$. Without loss of generality, we assume that it goes through the point B' in B's neighborhood. Use $\mathcal{P}^*_{CB'}$ and $\mathcal{P}^*_{B'D}$ to denote the two subpaths from C to B' and B' to D respectively. They must also be shortest paths of CB' and $B'D$.

The path $\mathcal{P}^*_{CB'}$ is at most ℓ_0 shorter than the shortest path between C and B, otherwise the path $\mathcal{P}^*_{CB'}$ concatenated with ring edges from B' to B would be shorter than the shortest path. Similarly, $\mathcal{P}^*_{B'D}$ is at most ℓ_0 shorter than the shortest path between B and D. Therefore the shortest path between C and D is at most $2\ell_0$ shorter than the concatenation of shortest paths of CB and BD. Then by Lemma 12, the summation of $\mathcal{P}^+_{CB}$ and $\mathcal{P}^-_{DB}$ is at most $2(3\ell_0 + 1) + 2\ell_0 = 8\ell_0 + 2$ longer than the shortest path between C and D. $\square$

We have the following corollary since the two paths in this lemma differ by at most 2 considering the length.

Corollary 14. *The concatenation of $\mathcal{P}^+_{CB}$ and reversed $\mathcal{P}^-_{DB}$, and the concatenation of reversed $\mathcal{P}^+_{AC}$ and $\mathcal{P}^-_{DA}$ are both at most $8\ell_0 + 4$ longer than the shortest path between C and D.*

Now we can prove the lower bound of δ for this case.

Proof of Theorem 1 for the case of $d = 1$ and $\gamma > 3$. Consider the four points A, B, C and D defined above. Let $d(x, y)$ denote the distance between vertices x and y. We can see the following consequences about pairwise distances: (a) $d(A, B) \geq \lfloor n/2 \rfloor / \ell_0$; (b) $d(C, D) \geq |\mathcal{P}^+_{CB}| + |\mathcal{P}^-_{DB}| - (8\ell_0 + 4) \geq |\mathcal{P}^+_{AC}| - 1 + |\mathcal{P}^-_{DB}| - (8\ell_0 + 4) \geq d(A, C) + d(D, B) - (8\ell_0 + 5)$, where the first inequality is due to Corollary 14; and (c) similarly, $d(C, D) \geq d(A, D) + d(C, B) - (8\ell_0 + 5)$. Therefore, we have both $d(A, B) + d(C, D) \geq d(A, C) + d(D, B) + \lfloor n/2 \rfloor / \ell_0 - (8\ell_0 + 5)$, and $d(A, B) + d(C, D) \geq d(A, D) + d(C, B) + \lfloor n/2 \rfloor / \ell_0 - (8\ell_0 + 5)$.

For $\ell_0 < n^{\frac{1}{\gamma-1}+\varepsilon}$ with any sufficiently small $\varepsilon > 0$ and sufficiently large n, we have $\lfloor n/2 \rfloor / \ell_0 >> 8\ell_0 + 5$, and thus $d(A, B) + d(C, D)$ is the largest distance pair. In this case, $\delta \geq \lfloor n/2 \rfloor / \ell_0 - (8\ell_0 + 5)$. By Lemma 10, with probability $1 - o(1)$ there is $\ell_0 < n^{\frac{1}{\gamma-1}+\varepsilon}$. Therefore, with probability $1 - o(1)$, $\delta(KSW(n, 1, \gamma)) = \Omega(n/n^{\frac{1}{\gamma-1}+\varepsilon}) = \Omega(n^{\frac{\gamma-2}{\gamma-1}-\varepsilon})$ for $d = 1$, $\gamma > 3$ and any sufficiently small $\varepsilon > 0$. Since for any $\varepsilon' > \varepsilon > 0$, $n^{\frac{\gamma-2}{\gamma-1}-\varepsilon} = \Omega(n^{\frac{\gamma-2}{\gamma-1}-\varepsilon'})$, we have $\delta(KSW(n, 1, \gamma)) = \Omega(n^{\frac{\gamma-2}{\gamma-1}-\varepsilon})$ for any $\varepsilon > 0$. $\square$

3.4 Extensions of the *KSW* model

Our analysis also holds for some variants of the *KSW* model. In this section, we study one variant of the underlying structure: grid without wrap-around; and two variants of edge linking: multiple edges for each vertex and linking edges independently.

Grid without wrap-around. We modify our analysis so that the first two results of Theorem 1 still hold. For the case of $d \geq 2$ and $0 \leq \gamma \leq d$ (Section 3.3.1), the changes are as follows. For a path from u to v, we divide it into segments as before. Elements in an edge vector are in $\{-n', -n' + 1, \ldots, n' - 1, n'\}$ now. Recall the last condition that we define two paths from u to v belong to the same category: the summations (not module n') of all segment vectors in the two paths are equal. This is always satisfied for grid without wrap-around, because the summation of all segment vectors depends only on the positions of u and v. In the proofs of Lemma 3 and Lemma 4, the summation of all segment vectors is fixed rather than $(2\ell + 1)^d$ or $(2k + 1)^d$ choices respectively. Hence the upper bounds given in Lemmas 3 and 4 still hold. In Lemma 5, an edge between u and v can be from u to v or from v to u. The probabilities of the two cases may differ by a constant factor on grid without wrap-around. Hence the probability of existing an edge between u and v is changed by at most a constant factor, and Lemma 5 still holds. One can verify the rest analysis in Section 3.3.1 still hold. For the case of $d = 1$ and $0 \leq \gamma \leq 1$ (Section 3.3.2), the only change is that event $\mathcal{E}_i$, which is the event that v_i links to $v_{i+\ell_0}$, only applies when $i = 0, 1, \ldots, n - \ell_0 - 1$. Since $\ell_0 = O(\log n)$ is much smaller than n, there are still almost n events $\mathcal{E}_i$ and the argument has no significant change. Hence Theorem 1 still holds for the cases that $d \geq 1$ and $0 \leq \gamma \leq d$.

Multiple edges for each vertex. In this model, each vertex links a constant, say d_0, number of edges according to the same distribution that u links to v with probability $\frac{d_B(u,v)^{-\gamma}}{\sum_{v'} d_B(u,v')^{-\gamma}}$. We show that all our analysis still hold with some slight changes. For the case of $d \geq 2$ and $1 \leq \gamma \leq d$ (Section 3.3.1), Lemma 5 still holds since the probability of the edge (u, v) is increased by at most d_0 times using union bound. And the proof of Lemma 6 still works, because $O(i^{d-1} \frac{i^{-\gamma}}{f(n')})$ is an upper bound of the probability that u links to some vertex at distance i by union bound. For the case of $d = 1$ and $0 \leq \gamma \leq 1$ (Section 3.3.2), we define $\mathcal{E}_i$ as the event that at least one of v_i's edges links to $v_{(i+\ell_0 \bmod n)}$. One can see $\Pr\{\mathcal{E}_i\}$ is still $\Theta(\frac{\ell_0^{-\gamma}}{f(n)})$ and the rest argument also holds. For the case of $d = 1$ and $\gamma > 3$ (Section 3.3.3), Lemma 10 still holds for the same reason as Lemma 6, that $O(i^{d-1} \frac{i^{-\gamma}}{\sum_{j=1}^{n^{1/d}} j^{d-1} j^{-\gamma}})$ is an upper bound of the probability that u links to some vertex at distance i. One can verify all results of Theorem 1 still hold under this change.

Linking edges independently. In this model, all edges exist independently. The edge between (u, v) exists with probability $\frac{d_0 \cdot d_B(u,v)^{-\gamma}}{\sum_{i=1}^{n^{1/d}} i^{d-1-\gamma}}$, where d_0 is some constant. We also give the changes in our analysis. Lemma 5 is straightforward in this model. Lemmas 6 and 10 still hold for the same reason as above. In the proof of Lemma 7, the first line of Eq (2), which uses the multiplication of edges' probabilities for an upper bound of the path's probability, still holds because it is now just the multiplication of independent events. For the case of $d = 1$ and $0 \leq \gamma \leq 1$ (Section 3.3.2), we define $\mathcal{E}_i$ to be the event that the edge

$(v_i, v_{(i+\ell_0 \bmod n)})$ exists, one can see the analysis still works. All results of Theorem 1 still hold under this change.

In summary, Theorem 1 of the case $d \geq 1$ and $0 \leq \gamma \leq d$ still holds for grid without wrap-around, and Theorem 1 of all cases still holds for both variants of edge linking. The variants of edge linking can be combined with grid without wrap-around, for which Theorem 1 of the case $d \geq 1$ and $0 \leq \gamma \leq d$ still holds.

4 δ-hyperbolicity of ringed trees

In this section, we consider the δ-hyperbolicity of graphs constructed according to a variant of the small-world graph model, in which long-rang edges are added on top of a base graph that is a binary tree or tree-like low-δ graph. In particular, we will analyze the effect on the δ-hyperbolicity of adding long-range links to a ringed tree base graph; and then we will consider several related extensions, including an extension to the binary tree.

Definition 6 (Ringed tree)**.** *A* ringed tree *of level k, denoted $RT(k)$, is a fully binary tree with k levels (counting the root as a level), in which all vertices at the same level are connected by a ring. More precisely, we can use a binary string to represent each vertex in the tree, such that the root (at level 0) is represented by an empty string, and the left child and the right child of a vertex with string σ are represented as $\sigma 0$ and $\sigma 1$, respectively. Then, at each level $i = 1, 2, \ldots, k-1$, we connect two vertices u and v represented by binary strings σ_u and σ_v if $(\sigma_u + 1) \mod 2^i = \sigma_v$, where the addition treats the binary strings as the integers they represent. As a convention, we say that a level is higher if it has a smaller level number and thus is closer to the root.*

Figure 1(d) illustrates the ringed tree $RT(6)$. Note that the diameter of the ringed tree $RT(k)$ is $\Theta(\log n)$, where $n = 2^k - 1$ is the number of vertices in $RT(k)$, and we will use $RT(\infty)$ to denote the infinite ringed tree when k in $RT(k)$ goes to infinity. Thus, a ringed tree may be thought of as a soft version of a binary tree; and to some extent, one can view a ringed tree as an idealized picture reflecting the hierarchical structure in real networks coupled with local neighborhood connections, such as Internet autonomous system (AS) networks, which has both a hierarchical structure of different level of AS'es, and peer connections based on geographical proximity.

4.1 Results and their implications

A visual comparison of the ringed tree of Figure 1(d) with the tessellation of Poincaré disk (Figure 1(b)) suggests that the ringed tree can been seen as an approximate tessellation or coarsening of the Poincaré disk. Our first result in this section makes this precise; in particular, we show that the infinite ringed tree and the Poincaré disk are quasi-isometric.

Theorem 2. *The infinite ringed tree $RT(\infty)$ and the Poincaré disk are quasi-isometric.*

Thus, by Proposition 1, we immediately have the following result.

Corollary 15. *There exists a constant c s.t., for all k, ringed tree $RT(k)$ is c-hyperbolic.*

Alternatively, we also provide a direct proof of this corollary (Section 4.3.3) to show that the ringed tree $RT(k)$ is Rips 5-hyperbolic, and Gromov's 40-hyperbolic in terms of the four point condition. Our direct analysis also provides important properties of ringed trees that are used by later analyses.

Next, we address the question of whether long-range edges added at each level of the ring maintains or destroys the hyperbolicity of the base graph. Given two vertices u and v at some level t of the ringed tree, we define the ring distance between u and v, denoted $d_R(u, v)$, to be the length of the shorter path connecting u and v purely through the ring edges at the level t. Given any function f from positive integers to positive integers, let $RT(k, f)$ denote the class of graphs constructed by adding long-range edges on the ringed tree $RT(k)$, such that for each long-range edge (u, v) connecting vertices u and v at the same level, $d_R(u, v) \leq f(n)$, where $n = 2^k - 1$ is the number of vertices in the ringed tree $RT(k)$. Since long-range edges do not reduce distances from root to any other vertices, the diameter of any graph in $RT(k, f)$ is still $\Theta(\log n)$. Define $\delta(RT(k, f)) = \max_{G \in RT(k,f)} \delta(G)$.

Our second result (used in the proof of the first part of our next result, but explicitly stated here since it is also of independent interest) is the following.

Theorem 3. *$\delta(RT(k, f)) = O(\log f(n))$, for any positive function f and positive integer k, where $n = 2^k - 1$ is the number of vertices in the ringed tree $RT(k)$.*

This result indicates that if the long-range edges added do not span far-away vertices, then the graph should have good hyperbolicity. In particular, if we take $f(n) = \log n$, then the theorem implies that the class $RT(k, f)$ is logarithmically hyperbolic. The theorem covers all (deterministic) graphs in the class $RT(k, f)$. We can extend it to random graphs, such that if we can show that with high probability the random graph is in the class $RT(k, f)$, then we know that the hyperbolic δ of the random graph is $O(\log f(n))$ with high probability. The first result in the next theorem is proven via this approach.

Next, we consider adding random edges between two vertices at the outermost level, i.e., level $k - 1$, such that the probability connecting two vertices u and v is determined by a function $g(u, v)$. Let V_{k-1} denote the set of vertices at level $k - 1$, i.e., the leaves of the original binary tree. Given a real-valued positive function $g(u, v)$, let $RRT(k, g)$ denote a random graph constructed as follows. We start with the ringed tree $RT(k)$, and then for each vertex $v \in V_{k-1}$, we add one long-range edge to a vertex u with probability proportional to $g(u, v)$, that is, with probability $g(u, v)\rho_v^{-1}$ where $\rho_v = \sum_{u \in V_{k-1}} g(u, v)$.

We study three families of functions g, each of which has the characteristic that vertices closer to one another (by some measure) are more likely to be connected by a long-range edge. The first two families use the ring distance $d_R(u, v)$ as the closeness measure. In particular, the first family uses an exponential decay function $g_1(u, v) = e^{-\alpha d_R(u,v)}$. The second family uses a power-law decay function $g_2(u, v) = d_R(u, v)^{-\alpha}$, where $\alpha > 0$. The third family uses the height of the lowest common ancestor of u and v, denoted as $h(u, v)$, as the closeness measure, and the function is $g_3 = 2^{-\alpha h(u,v)}$. Note that this last probability function matches the function used by Kleinberg in a small-world model based on the tree structure [23]. The following theorem summarizes the hyperbolicity behavior of these three families of random ringed trees.

Theorem 4. *Considering the follow families of functions (with u and v as the variables of the function) for random ringed trees $RRT(k, g)$, for any positive integer k and positive real number α, with probability $1 - o(1)$ (when n tends to infinity), we have*

1. *$\delta(RRT(k, e^{-\alpha d_R(u,v)})) = O(\log \log n)$;*
2. *$\delta(RRT(k, d_R(u, v)^{-\alpha})) = \Theta(\log n)$;*
3. *$\delta(RRT(k, 2^{-\alpha h(u,v)})) = \Theta(\log n)$;*

where $n = 2^k - 1$ is the number of vertices in the ringed tree $RT(k)$.

This theorem states that, when the random long-range edges are selected using exponential decay function based on the ring distance measure, the resulting graph is logarithmically hyperbolic, i.e., the constant hyperbolicity of the original base graph is degraded only slightly; but when a power-law decay function based on the ring distance measure or an exponential decay function based on common ancestor measure is used, then hyperbolicity is destroyed and the resulting graph is not hyperbolic. One may notice that the function form in (1) and (3) above is similar but the result is different. This is because with height $h(u, v)$ the subtree covers actually $\Theta(2^{h(u,v)})$ leaves, and thus (3) is naturally closer to the power-law function of (2). Intuitively, when it is more likely for a long-range edge to connect two far-away vertices, such an edge creates a shortcut for many internal tree nodes so that many shortest paths will go through this shortcut instead of traversing through tree nodes. (In Internet routing this is referred to as *valley routes*).

Finally, as a comparison, we also study the hyperbolicity of random binary trees $RBT(k, g)$, which is the same as random ringed trees $RRT(k, g)$ except that we remove all ring edges.

Theorem 5. *Considering the follow families of functions (with u and v as the variables of the function) for random binary trees $RBT(k, g)$, for any positive integer k and positive real number α, with probability $1 - o(1)$ (when n tends to infinity), we have*

$$\delta(RBT(k, e^{-\alpha d_R(u,v)})) = \delta(RBT(k, d_R(u, v)^{-\alpha})) = \delta(RBT(k, 2^{-\alpha h(u,v)})) = \Theta(\log n),$$

where $n = 2^k - 1$ is the number of vertices in the binary tree $RBT(k, g)$.

Thus, in this case, the original hyperbolicity of the base graph ($\delta = 0$ for the binary tree) is destroyed. Comparing with Theorem 4, our results above suggest that the "softening" of the hyperbolicity provided by the rings is essential in maintaining good hyperbolicity: with rings, random ringed trees with exponential decay function (depending on the ringed distance) are logarithmically hyperbolic, but without the rings the resulting graphs are not hyperbolic.

4.2 Outline of the analysis

In this subsection, we provide a summary of the proof of the four theorems in Section 4.1. For Theorem 2, we provide an embedding of the ringed tree to the Poincaré disk, intuitively similar to the picture we show in Figure 1(d), and prove that it is a quasi-isometry.

For the analysis of δ-hyperbolicity, we apply the Rips condition, which is equivalent to the Gromov's four point condition up to a constant factor. For any two vertices u and v on the ringed tree $RT(k)$, we define the canonical geodesic $\langle u, v\rangle$ to be the geodesic from u to v such that the geodesic always goes up first, then follows ring edges, and then goes down (any of these segments may be omitted). We show that the canonical geodesic $\langle u, v\rangle$ and any other geodesic $[u, v]$ are within distance 1 of each other, and any triangle $\Delta(u, v, w)$ formed by three canonical geodesics $\langle u, v\rangle$, $\langle u, w\rangle$, and $\langle v, w\rangle$ (called *canonical triangle*) are 3-slim. This immediately implies that any geodesic triangles in $RT(k)$ is 5-slim, which is a direct proof that ringed trees are constantly hyperbolic.

For Theorem 3, we inductively prove that any geodesic $[u, v]$ in $RT(k, f)$ is within $O(\log f(n))$ distance from the canonical geodesic $\langle u, v\rangle$, and vice versa. Together with the result that any canonical triangle is 3-slim, it follows know that any geodesic triangle is $O(\log f(n))$-slim. For Theorem 4, Part (1), we show that with high probability the long-range edges only connect vertices within ring distance $O(\log n)$, and then we can apply Theorem 3 to achieve the $O(\log \log n)$ bound. For Theorem 4, Part (2), the key is to show that (a) with high probability some long-range link connects two vertices at ring distance $\Theta(n^c)$ for some constant c; and (b) if such a long-range edge (u, v) exists, then we consider the geodesic triangle $\Delta(u, v, r)$ where r

is a point with lowest layer number on the canonical geodesic between u, v, and show that the middle point of $[r, u]$ is $\Theta(\log n)$ away from the union of $[r, v]$ and $[v, u]$. For Theorem 4, Part (3), we first show that with high probability some pair of vertices u, v with $h(u, v) \geq c \log_2 n$ for some constant $c > 0$, and then we observe that, in such configuration, the ring distance $d_R(u, v)$ has high probability to be $\Omega(n^{c/2})$, and results follows exactly the same analysis in the previous part.

For Theorem 5, part (2) and (3) follow a similar strategy as those of Theorem 4. For part (1), we know that two "would-be" ring neighbors u and v have constant probability of having a long-range connection. However, since we do not have ring edges, the alternative path between u and v through the tree may be $\Theta(\log n)$ in length. We show that there are at least $\Omega(\sqrt{n})$ such pairs, so with high probability at least one pair is connected, generating a bad δ of $\Omega(\log n)$.

4.3 Detailed analysis on ringed trees

4.3.1 Properties of ringed tree

We start by some properties of ringed tree, which will be repeatedly used in the following analysis on ringed tree related graphs and which may be of independent interest.

We define the *ring distance* $d_R(u, v)$ of u and v on the same level to be their distance on the ring. Ringed trees have the following fundamental property.

Lemma 16. *Let u and v be two vertices on the same level, and u' and v' be their parents respectively. We have $d_R(u', v') \leq (d_R(u, v) + 1)/2$.*

Proof. On the ring, there are $d_R(u, v) + 1$ vertices on segment between u and v, which belong to at most $(d_R(u, v)+1)/2+1$ parents, which correspond to at most $(d_R(u, v)+1)/2+1$ vertices on the ring segment between u' and v'. This concludes the proof. □

For a geodesic $[u, v]$ on the ringed tree $RT(k)$, we call its *level sequence* the sequence of levels it passes by from u to v. Lemma 16 implies that the level sequence of any geodesic must be reversed unimodal: it first decreases, and then increases (but the increasing or decreasing segment may be omitted). The following lemma further characterizes geodesics in ringed-trees.

Lemma 17. *Let u, v be two vertices, u be on level ℓ, and u' be the parent of u at level $\ell - 1$. Suppose $[u, v]$ intersects level $\ell - 1$, and let t be the intersection closest to u. Then $d(u, t) \leq 2$, the segment $[u, t]$ of $[u, v]$ and $\{u, u'\}$ are within distance 1 to each other.*

Proof. Let t' be the node just before t to u on $[t, u]$. $d(t, u) \leq 1 + d_R(t, u') \leq 1 + (d_R(t', u) + 1)/2$ by Lemma 16. As t is the closest node on level $\ell - 1$ on the geodesic to u, $d(t, u) \geq 1 + d_R(t', u)$. We get $d(t', u) \leq 1$ and $d(t, u) \leq 2$ by combining these two inequalities. The segment $[u, t]$ of $[u, v]$ and $\{u, u'\}$ are within distance 1 to each other since $d(t, u') \leq 1$. □

For two vertices u and v, we define the *canonical geodesic* $\langle u, v \rangle$ in a recursive fashion.

1. For u, v on the same level and $d_R(u, v) \leq 3$, $\langle u, v \rangle$ is the path on the ring from u to v.

2. For u, v on the same level but $d_R(u, v) > 3$, let u', v' be parents of u, v respectively, then $\langle u, v \rangle = [u, u'] \cup \langle u', v' \rangle \cup [v', v]$.

3. For u, v on different levels, supposing u on upper level, let v' be parent of v, then $\langle u, v \rangle = \langle u, v' \rangle \cup [v', v]$.

This is a well-founded definition. At each level of recursion, either the difference of levels of nodes decreases, or in the case of nodes on the same level, ring distance decreases by Lemma 16, until we reach the base case, where $d_R(u, v) \leq 3$.

We prove now that canonical geodesics are really geodesics.

Lemma 18. *For any u, v, $\langle u, v\rangle$ is a geodesic between u, v.*

Proof. For u, v on the same level and $d_R(u, v) \leq 3$, we can check that $\langle u, v\rangle$ is a geodesic between u, v.

For u, v on the same level but $d_R(u, v) > 3$, let u', v' be parents of u, v respectively. Let t, s be the closest node to u, v to be in an upper level on $[u, v]$ respectively. If $t \neq u'$, then $d(u, t) \geq 2$ because the only way to go up one level in one step is to go to the parent. By Lemma 17, $[u, t]$ and $\{u, u'\}$ are within distance 1 to each other. Therefore $d(u', t) = 1$, as $d(u, t) \geq 2$, and $\{u, u'\} \cup [t, v]$ is also a geodesic from u to v. This is also correct for $t = u'$. The same can be proved for s. By combining, we have that $\{u, u'\} \cup [t, s] \cup \{v', v\}$ is also a geodesic between u, v for any geodesics $[t, s]$. If we pick the canonical geodesic $\langle t, s\rangle$, in any cases, this will be the canonical geodesic $\langle u, v\rangle$. Therefore, $\langle u, v\rangle$ is a geodesic.

For u, v on different levels, the induction is essentially the same as in the previous case, but we only need to reason on t only on the side of u.

This concludes our induction. □

4.3.2 Proof of Theorem 2: Quasi-isometry from infinite ringed tree to the Poincaré disk

In this subsection, we will exhibit and prove a quasi-isometry from the infinite ringed tree $RT(\infty)$ to the Poincaré disk. We denote its distance d_{RT}. We denote (D, d_P) the Poincaré disk, where D is the open disk of radius 1 on the complex plane.

Here is a brief summary of our approach here. First we propose a candidate of quasi-isometry, then all possible cases of images of two points of ringed tree are divided into four categories, each of which is separately analyzed. We then proceed with an analysis on the metric of ringed tree and show that the candidate of quasi-isometry is effective, thus ringed tree and the Poincaré disk are quasi-isometric.

The following inequalities are used in the following.

$$\ln(x) \leq \cosh^{-1}(x) \leq ln(2x) \quad \text{for} \quad x \geq 1$$

$$\frac{2}{\pi}x \leq \sin(x) \leq x \quad \text{for} \quad 0 \leq x \leq \frac{\pi}{2}$$

$$1 - \frac{x}{2} \leq \sqrt{1-x} \leq 1 - \frac{3x}{5} \quad \text{for} \quad 0 \leq x \leq \frac{1}{2}$$

To state the promised quasi-isometry, we give coordinates to nodes in $RT(\infty)$. We know that we can number nodes with binary strings, which can be regarded as a number. For a node on the k-th level and numbered by m, its coordinates are (k, m), with $0 \leq m \leq 2^k - 1$. The root is level 0.

Definition 7. *Let the following mapping be the candidate of quasi-isometry :*

$$f : RT(\infty) \to D, (k, m) \mapsto \sqrt{1 - 2^{-k}}e^{2i\pi\frac{m}{2^k}}$$

For $0 \leq k \leq \ell$, $m < 2^{\ell-1}$, we define $D(k, \ell, m) = d_P(f(k, 0), f(\ell, m))$. This is a distance in the Poincaré disk. Following is its full expression in k and m.

$$\begin{aligned} D(k, \ell, m) &= \cosh^{-1}\left(1 + 2\frac{\|\sqrt{1-2^{-k}} - \sqrt{1-2^{-\ell}}e^{2i\pi\frac{m}{2^\ell}}\|^2}{2^{-k-\ell}}\right) \\ &= \cosh^{-1}\left(1 + 2\left(\sqrt{2^\ell(2^k-1)} - \sqrt{2^k(2^\ell-1)}\right)^2 + 8\sqrt{2^\ell(2^\ell-1)2^k(2^k-1)}\sin^2\left(\pi\frac{m}{2^\ell}\right)\right) \end{aligned}$$

We also define $D'(k,\ell,m) = d_{RT}(f(k,0), f(\ell,m))$, with $0 \leq k \leq \ell$, $m < 2^{\ell-1}$. This is a distance in ringed tree.

We will now try to bound $D(k,\ell,m)$ with the following lemma.

Lemma 19. *We have the following bounds on $D(k,\ell,m)$.*

1. *For $k = 0$,*
$$\frac{\ln(2)}{2}\ell + \frac{\ln(2)}{2} \leq D(0,\ell,m) \leq \frac{\ln(2)}{2}\ell + \ln(6).$$

2. *For $0 < k = \ell, m > 0$,*
$$\ln(2)(4 + 2\lfloor \log_2 m \rfloor) \leq D(k,k,m) \leq \ln(2)(4 + 2\lfloor \log_2 m \rfloor) + \ln(\frac{5}{4\pi^2})\text{for}(k \geq 1).$$

3. *For $m = 0, 1 \leq k < \ell$,*
$$\ln(2)(\ell - k) - \ln(50) \leq D(k,\ell,0) \leq \ln(2)(\ell - k).$$

4. *For $0 < k < \ell$ and $0 < m < 2^{\ell-k}$,*
$$\ln(2)(\ell - k) - \ln(100) \leq D(k,\ell,m) \leq \ln(2)(\ell - k) + \ln(66\pi^2).$$

5. *For $0 < k < \ell$ and $2^{\ell-k} \leq m < 2^{\ell-1}$,*
$$\ln(2)(k - \ell + 2\lfloor \log_2 m \rfloor + 4) \leq D(k,\ell,m) \leq \ln(2)(k - \ell + 2\lfloor \log_2 m \rfloor + 6) + \ln(\pi^2 + 1).$$

Proof. 1. Case $k = 0$
$$D(0,\ell,m) = \cosh^{-1}(1 + 2\sqrt{2^\ell - 1})$$
If $\ell = 0$, $D(0,0,m) = 0$. For $\ell \geq 1$, we have
$$D(0,\ell,m) \geq \cosh^{-1}(2\sqrt{2^{\ell-1}}) \geq \frac{\ln(2)}{2}\ell + \frac{\ln(2)}{2}$$
and
$$D(0,\ell,m) \leq \cosh^{-1}(3\sqrt{2^\ell}) \leq \frac{\ln(2)}{2}\ell + \ln(6).$$

2. Case $0 < k = \ell, m > 0$
$$D(k,k,m) = \cosh^{-1}(1 + 2^{k+3}(2^k - 1)\sin^2(\frac{m}{2^k}\pi))$$
Let $a = \lfloor \log_2 m \rfloor$, we have
$$D(k,k,m) \geq \cosh^{-1}(2^{2k+2}\sin^2(2^{a-k}\pi)) \geq \cosh^{-1}(2^{2k+2}(2^{a+1-k})^2) \geq \ln(2)(4 + 2a)$$
and
$$\begin{aligned} D(k,k,m) &\leq \cosh^{-1}(\frac{5}{4}2^{2k+2}\sin^2(2^{a+1-k}\pi)) \leq \cosh^{-1}(\frac{5}{4}2^{2k+2}(2^{a+1-k}\pi)^2) \\ &\leq \ln(2)(4 + 2a) + \ln(\frac{5}{4\pi^2}) \end{aligned}$$

3. Case $0 < k < \ell, m = 0$

$$\begin{aligned} D(k,\ell,0) &= \cosh^{-1}(1 + 2(\sqrt{2^\ell(2^k-1)} - \sqrt{2^k(2^\ell-1)})^2) \\ &= \cosh^{-1}(1 + 2^{k+\ell+1}(\sqrt{1-2^{-k}} - \sqrt{1-2^{-\ell}})^2) \end{aligned}$$

As $1 \leq k < \ell$, $\sqrt{1-2^{-\ell}} - \sqrt{1-2^{-k}} > 0$, and we have

$$\sqrt{1-2^{-\ell}} - \sqrt{1-2^{-k}} \geq 1 - 2^{-\ell-1} - 1 + \frac{3}{5}2^{-k} \geq \frac{1}{5}2^{-k-1}$$

and

$$\sqrt{1-2^{-\ell}} - \sqrt{1-2^{-k}} \leq 1 - \frac{3}{5}2^{-\ell} - 1 + 2^{-k-1} \leq 2^{-k-1}$$

Therefore, we have $\ln(2)(\ell-k) - \ln(50) \leq D(k,\ell,0) \leq \ln(2)(\ell-k)$.

4. Case $0 < k < \ell, 0 < m < 2^{\ell-1}$

We now deal with the general case with $m > 0$ and $0 < k < \ell$.

$$D(k,\ell,m) = \cosh^{-1}(\cosh(D(k,\ell,0)) + 8\sqrt{2^\ell(2^\ell-1)2^k(2^k-1)}\sin^2(\pi\frac{m}{2^\ell}))$$

We always note $a = \lfloor \log_2 m \rfloor$, and we have

$$\sqrt{2^\ell(2^\ell-1)2^k(2^k-1)}\sin^2(\pi\frac{m}{2^\ell}) \geq 2^{\ell+k-1}2^{2a-2\ell+2} = 2^{k-\ell+2a+1}$$

and

$$\sqrt{2^\ell(2^\ell-1)2^k(2^k-1)}\sin^2(\pi\frac{m}{2^\ell}) \leq 2^{\ell+k}2^{2a+2-2\ell}\pi^2 \leq 2^{k-\ell+2a+2}\pi^2$$

The previous bound on $D(k,\ell,0)$ transforms into the following by applying $\cosh$.

$$\frac{2^{\ell-k}}{100} \leq \cosh(D(k,\ell,0)) \leq 2^{\ell-k}$$

Suitable substitution of $\cosh(D(k,\ell,0))$ gives

$$\cosh^{-1}(\frac{2^{\ell-k}}{100} + 2^{k-\ell+2a+4}) \leq D(k,\ell,m) \leq \cosh^{-1}(2^{\ell-k} + 2^{k-\ell+2a+5}\pi^2).$$

For $0 \leq a \leq \ell-k$, therefore $0 < m < 2^{\ell-k}$,

$$D(k,\ell,m) \geq \cosh^{-1}(\frac{2^{\ell-k}}{100} + 2^{k-\ell+2a+4}) \geq \ln(\frac{2^{\ell-k}}{100}) = \ln(2)(\ell-k) - \ln(100)$$

and

$$\begin{aligned} D(k,\ell,m) &\leq \cosh^{-1}(2^{\ell-k} + 2^{k-\ell+2a+5}\pi^2) \\ &\leq \ln(2^{\ell-k}33\pi^2) + ln(2) = \ln(2)(\ell-k+1) + \ln(33\pi^2). \end{aligned}$$

For $\ell-k+1 \leq a < m-1$, therefore $2^{\ell-k} \leq m < 2^{\ell-1}$,

$$D(k,\ell,m) \geq \cosh^{-1}(\frac{2^{\ell-k}}{100} + 2^{k-\ell+2a+4}) \geq \ln(2^{k-\ell+2a+4}) = \ln(2)(k-\ell+2a+4)$$

and

$$
\begin{aligned}
D(k,\ell,m) &\leq \cosh^{-1}(2^{\ell-k}+2^{k-\ell+2a+5}\pi^2) \\
&\leq \ln((\pi^2+1)2^{k-\ell+2a+5})+ln(2)=\ln(2)(k-\ell+2a+6)+\ln(\pi^2+1).
\end{aligned}
$$

□

We will now try to relate $D(k,\ell,m)$ and $D'(k,\ell,m)$ in the following lemma.

Lemma 20. *For any* $0 \leq k \leq \ell, 0 \leq m < 2^{\ell-1}$,

$$\frac{\ln(2)}{2}D'(k,\ell,m)-\ln(200) \leq D(k,\ell,m) \leq \ln(2)D'(k,\ell,m)+\ln(66\pi^2)$$

Proof. Consider the canonical geodesic. If $k=\ell$ and $m=1$, it is an edge from $(k,0)$ to $(k,1)$. In other cases, it goes up first from (ℓ,m) to a certain ancestor, then makes 2 or 3 moves on the ring, and finished by going straight down to $(k,0)$.

In the first case, $D'(k,k,1)=1$. In the second case, as the ancestor of any node (k,m) is $(k-1,\lfloor\frac{m}{2}\rfloor)$, by the form of canonical geodesics, we should first go up $\ell-k$ steps to reach level k. If $m \leq 2^{\ell-k+1}$, it reaches $(k,0)$ by at most an extra step. In this case, $\ell-k \leq D'(k,\ell,m) \leq \ell-k+1$. If $m > 2^{\ell-k+1}$, we go up $\lfloor\log_2 m\rfloor-1$ steps from (ℓ,m) to reach an ancestor numbered 2 or 3, then go 2 or 3 steps on the ring to the node numbered 0 on the same level, and finish by going down to $(k,0)$. Thus we have $2\lfloor\log_2 m\rfloor+k-\ell \leq D'(k,\ell,m)2\lfloor\log_2 m\rfloor+k-\ell+1$ in this case.

We conclude by comparing to bounds in Lemma 19. □

We want to bound distance between any two points in $RT(\infty)$ with the following lemma.

Lemma 21. *For* $u_1,v_1,u_2,v_2 \in RT(\infty)$ *with* u_1,v_1,u_2,v_2 *on the same level and* $d_R(u_1,v_1)=d_R(u_2,v_2)$, *we have* $|d(u_1,v_1)-d(u_2,v_2)| \leq 3$.

Proof. Let $A=d_R(u_1,v_1)$. It is clearly correct when $A \leq 3$ by the structure of canonical geodesics.

For $A \leq 3$, when we consider the canonical geodesics, we take successive ancestors of u_1 and v_1 until their distance is 2 or 3. This takes at least $\lfloor\log_2 A\rfloor-1$ generations, but at most $\lfloor\log_2(A-1)\rfloor$. These bounds differ by at most 1. Platforms differ by at most 1, therefore $|d(u_1,v_1)-d(u_2,v_2)| \leq 3$. □

Proof of Theorem 2. By symmetry, Lemma 21 and Lemma 20, for any $u,v \in Rt(\infty)$, we have:

$$\frac{\ln(2)}{2}d_{RT}(u,v)-\ln(200) \leq d_P(f(u),f(v)) \leq \ln(2)d_{RT}(u,v)+\ln(66\pi^2)$$

There is only one thing left to prove quasi-isometry. We will now prove that for some constant ϵ, $B(f(RT(\infty)),\epsilon)$ covers the Poincaré disk. Images of each level are all on concentric circles, and the difference of radius between successive levels can be bounded by $\ln(6)$. Distance between images of neighboring nodes on the same level can be bounded by $\ln(16)$. For any point in the Poincaré disk, its distance to the nearest image of nodes is bounded by $\ln(96)$, by first moving straight away from 0 until reaching a concentric circle of images, then take the shortest path to reach image of a certain node.

This proves f to be a $(\frac{2}{\ln(2)},\ln(66\pi^2))$-quasi-isometry from $RT(\infty)$ to the Poincaré disk, thus $RT(\infty)$ and the Poincaré disk are quasi-isometric. Constants we found here are not tight. □

4.3.3 A direct proof of Corollary 15

We begin with a lemma about the distance between a general geodesic and the corresponding canonical geodesic.

Lemma 22. *For any geodesic $[u, v]$, $[u, v]$ and $\langle u, v\rangle$ are within distance 1 to each other.*

Proof. We perform an induction on the structure of $\langle u, v\rangle$.

For u, v on the same level and $d_R(u, v) \leq 3$, we can check that $[u, v]$ and $\langle u, v\rangle$ are within distance 1 to each other.

For u, v on the same level but $d_R(u, v) > 3$, let u', v' be parents of u, v respectively. We have $\langle u, v\rangle = \langle u, u'\rangle \cup \langle u', v'\rangle \cup \langle v', v\rangle$ by construction. By Lemma 17, the parts of $[u, v]$ that are on the same level with u, v verify the condition already. We only need to deal with the part on levels with lower numbering.

Let t, s be the closest node to u, v that are in an upper level on $[u, v]$ respectively. If $t \neq u'$, then $d(u, t) \geq 2$ because the only way to go up one level in one step is to go to the parent. By Lemma 17, $[u, t]$ and $\langle u, u'\rangle$ are within distance 1 to each other. Therefore $d(u', t) = 1$, as $d(u, t) \geq 2$, $\langle u, u'\rangle \cup \langle u', t\rangle \cup [t, v]$ is also a geodesic from u to v. This is also correct for $t = u'$. The same can be proven for s. By combining the above, we have that $\langle u, u'\rangle \cup \langle u', t\rangle \cup [t, s] \cup \langle s, v'\rangle \cup \langle v', v\rangle$ is also a geodesic between u, v. Thus the section $\langle u', t\rangle \cup [t, s] \cup \langle s, v'\rangle$ is also a geodesic from u' to v'. By induction hypothesis, it is within distance 1 to $\langle u', v'\rangle$. However, this geodesic contains the part of $[u, v]$ on levels with lower numbering than that of u, v, which is precisely $[t, s]$. We conclude that $[u, v]$ and $\langle u, v\rangle$ are within distance 1 to each other.

For u, v on different levels, the induction is essentially the same as in the previous case, but we only need to reason on t on the side of u.

This concludes our induction. □

Lemma 23. *Let u and v be two vertices at the same level ℓ such that $\langle u, v\rangle$ stays at level ℓ. Then for any vertex w, $\langle u, w\rangle$ and $\langle v, w\rangle$ are within distance 3 of each other.*

Proof. Let ℓ_0 and ℓ'_0 be the highest levels reached by $\langle u, w\rangle$ and $\langle v, w\rangle$, respectively. Without loss of generality, we assume that $\ell_0 \geq \ell'_0$. Thus $\ell \geq \ell_0$. We prove the lemma by an induction on ℓ.

Consider the base case of $\ell = \ell_0$. Let w' be the ancestor of w at level ℓ_0. Thus we know that both $\langle w', u\rangle$ and $\langle u, v\rangle$ stay at level ℓ_0. By definition $d_R(w', u) \leq 3$ and $d_R(u, v) \leq 3$. We can enumerate all the possible cases of u, v, w' arrangement to check that $\langle w', v\rangle$ is at most one level higher than ℓ_0 (i.e., $\ell'_0 \geq \ell_0 - 1$), and $\langle w', v\rangle$ and $\langle w', u\rangle$ are within distance 3 of each other. Since $\langle u, w\rangle$ and $\langle v, w\rangle$ share the portion $\langle w, w'\rangle$, the lemma holds for the case of $\ell = \ell_0$.

For the induction step, consider $\ell > \ell_0$. Let u' and v' be the parents of u and v, respectively. By Lemma 16 and $d_R(u, v) \leq 3$, we know that $d_R(u', v') \leq 2$. Then $\langle u', v'\rangle$ must stay at level $\ell - 1$. By induction hypothesis, $\langle u', w\rangle$ and $\langle v', w\rangle$ are within distance 3 of each other. Since $\langle u, w\rangle = \langle u, u'\rangle \cup \langle u', w\rangle$, $\langle v, w\rangle = \langle v, v'\rangle \cup \langle v', w\rangle$, and $d_R(u, v) \leq 3$, we know that the lemma holds in this case. □

We define *canonical triangle* $\hat{\Delta}(u, v, w)$ to be a geodesic triangle in which all sides are canonical geodesics.

Lemma 24. *Any canonical triangle $\hat{\Delta}(u, v, w)$ in $RT(k)$ is 3-slim.*

Proof. Without loss of generality, let u be the vertex at the lowest level, which is ℓ. We prove the lemma by an induction on ℓ. The base case of $\ell = 0$ is trivial.

Consider the induction step with $\ell > 0$. If neither v nor w is at level ℓ, then both $\langle u, v\rangle$ and $\langle u, w\rangle$ go through u's parent u'. By induction hypothesis, we know that $\hat{\Delta}(u', v, w)$ is 3-slim. Adding u in this case

does not change the distance among the sides, and thus $\hat{\Delta}(u, v, w)$ is also 3-slim. Suppose now that v or w or both are at level ℓ. In the first case, suppose that at least one pair, say u and v, is such that $\langle u, v\rangle$ stays at level ℓ. By Lemma 23, we know that $\langle u, w\rangle$ and $\langle v, w\rangle$ are within distance 3 of each other. Since $\langle u, v\rangle$ has length at most 3, we know that $\hat{\Delta}(u, v, w)$ is 3-slim. In the second case, suppose that all pairs at level ℓ have their canonical geodesics into level $\ell - 1$. For each vertex at level ℓ, we take its parent, together with perhaps another vertex already within level $\ell - 1$, we can apply the induction hypothesis and show that their canonical triangle is 3-slim. Since for every vertex at level ℓ, its canonical geodesics to the other two vertices all go through its parent, the vertices at level ℓ do not change the distance between any pair of sides of the canonical triangle. Therefore, $\hat{\Delta}(u, v, w)$ is 3-slim. □

With Lemmas 22 and 24, we can provide a direct proof of Corollary 15.

Direct proof of Corollary 15. We use Rips condition here. For any u, v, w, By Lemma 22, we have $[u, v] \subseteq B(\langle u, v\rangle, 1)$. By Lemma 24, we have $\langle u, v\rangle \subseteq B(\langle u, w\rangle \cup \langle v, w\rangle, 3)$. By Lemma 22 again, we have $\langle u, w\rangle \cup \langle v, w\rangle \subseteq B([u, w] \cup [v, w], 1)$. Therefore, we conclude that $[u, v] \subseteq B([u, w] \cup [v, w], 5)$, and thus $\delta_{Rips}(RT(k)) \leq 5$. Since $\delta(RT(k)) \leq 8\delta_{Rips}(RT(k))$, we have $\delta(RT(k)) \leq 40$. □

4.3.4 Proof of Theorem 3

We now apply Rips condition to analyze the hyperbolicity of ringed trees with limited long-range edges $RT(k, f)$. First we show that a geodesic in $RT(k, f)$ cannot make too many hops at the same level of the ringed tree. For any two vertices in a graph $G \in RT(k, f)$, let $[u, v]$ denote any one of the geodesics between u and v in G, while $\langle u, v\rangle$ still denote the canonical geodesic between u and v in the base ringed tree $RT(k)$.

Lemma 25. *For any u, v on the same level in $RT(k)$ with $d(u, v) > 1$, $2\log_2 d_R(u, v) \leq d(u, v) \leq 2\log_2(d_R(u, v) - 1) + 2$.*

Proof. We perform induction on $d_R(u, v)$. If $1 < d_R(u, v) \leq 3$, we can check that it is correct. If $d_R(u, v) > 3$, let u', v' be parents of u, v respectively. By induction hypothesis, $2\log_2 d_R(u', v') \leq d(u', v') \leq 2\log_2(d_R(u', v') - 1) + 2$. By triangle inequality, $d(u, v) \leq d(u', v') + 2 \leq 2\log_2(d_R(u', v') - 1) + 4$. But $d_R(u', v') - 1 \leq (d_R(u, v) - 1)/2$ by Lemma 16. And we have $d(u, v) \leq 2\log_2(d_R(u', v') - 1) + 4 \leq 2\log_2(d_R(u, v) - 1) + 2$. On the other hand, $d(u, v) \geq 2 + d(u', v') \geq 2\log_2(2d_R(u', v'))$. As $d_R(u', v') \geq d_R(u, v)/2$, we have $d(u, v) \geq 2\log_2 d_R(u, v)$. Combining this two inequalities concludes the induction. □

Corollary 26. *For any u, v on the same level in $RT(k)$, $d(u, v) \leq 2\log_2 d_R(u, v) + 2$.*

Proof. We check for the case $d(u, v) = 1$ and it is satisfied. For $d(u, v) > 1$, this results directly from $d(u, v) \leq 2\log_2(d_R(u, v) - 1) + 2 \leq 2\log_2 d_R(u, v) + 2$. □

Lemma 27. *For a graph in the class $RT(k, f)$ and two vertices u and v at the same level j, if $[u, v]$ never goes into vertices at level $i < j$, then $d(u, v) \leq \max(32, 4\log_2 f(n)) - 1$ and $d_R(u, v) \leq f(n)\max(32, 4\log_2 f(n)) - 1$.*

Proof. First, we will prove $d_R(u, v)/f(n) \leq d(u, v) \leq 2\log_2 d_R(u, v) + 2$. The first inequality is because the shortest distance from u to v without going into vertices in level $i < j$ in the base ringed tree $RT(k)$ is $d_R(u, v)$, and each long-range edge can jump at most $f(n)$ hops on the ring at any level. The second inequality results directly from Corollary 26. Let $x = d(u, v)$. From the above two inequalities, we have

$x \leq 2\log_2 xf(n) + 2$. Then, $f(n) \geq 2^{(x-2)/2}/x \geq 2^{(x+1)/4}$ for $x \geq 32$, and thus $x = d(u,v) \leq \max(32, 4\log_2 f(n)) - 1$. It follows that $d_R(u,v) \leq f(n)\max(32, 4\log_2 f(n)) - 1$. □

Lemma 28. *For a graph in the class $RT(k, f(n))$ and two vertices u and v, $[u,v]$ and $\langle u,v\rangle$ are within distance $2\max(32, 4\log_2 f(n))$ to each other.*

Proof. The case of u and v are ancestor and descendant to each other on the tree are trivial. Thus we consider the case that u and v are not ancestor and descendant to each other. Let $\ell_0^{\langle\rangle}$ and $\ell_0^{[]}$ be the innermost level that geodesics $\langle u,v\rangle$ and $[u,v]$ reach, respectively. If $\ell_0^{[]} < \ell_0^{\langle\rangle}$, then $[u,v]$ uses at least two more tree edges than $\langle u,v\rangle$. This means $\langle u,v\rangle$ uses at least two ring edges. If $\langle u,v\rangle$ uses exactly two ring edges, all edges in $[u,v]$ must be tree edges, which is impossible to be a geodesic given that u and v are not ancestor and descendant to each other. Thus, $\langle u,v\rangle$ uses exactly three ring edges. Then $[u,v]$ uses exactly one ring or long-range edge. Let u' and v' be the ancestors of u and v at level $\ell_0^{\langle\rangle}$, respectively. In this case, the only possible situation is : (a) $\ell_0^{[]} = \ell_0^{\langle\rangle} - 1$, and (b) the parents of u' and v' are connected either by a ring edge or a long-range edge, which is used by $[u,v]$. Hence, $[u,v]$ and $\langle u,v\rangle$ share the tree edges from u to u' and v' to v, and $\langle u,v\rangle$ goes through ring edges from u' to v' while $[u,v]$ goes through the edge connecting the parents of u' and v'. We thus have that $[u,v]$ and $\langle u,v\rangle$ are within distance 1 of each other. Therefore, from now on, we consider $\ell_0^{[]} \geq \ell_0^{\langle\rangle}$.

Let ℓ_u and ℓ_v be the levels of u and v respectively. Without loss of generality, We can suppose that $\ell_u \geq \ell_v \geq \ell_0^{[]}$.

For any level ℓ both reachable by $[u,v]$ and $\langle u,v\rangle$, i.e., $\ell_0^{[]} \leq \ell \leq \ell_u$, let x_ℓ be the first level-ℓ vertex on geodesic $[u,v]$ starting from u, and let y_ℓ be the first level-ℓ vertex on geodesic $\langle u,v\rangle$ starting from u. We claim that $d_R(x_\ell, y_\ell) \leq f(n)\max(32, 4\log_2 f(n))$.

For level ℓ_u, it is trivial. Suppose that our claim is correct for some level $\ell > \ell_0^{[]}$, and we inductively prove the claim for level $\ell-1$. By the induction hypothesis, we know that $d_R(x_\ell, y_\ell) \leq f(n)\max(32, 4\log_2 f(n))$. Let x'_ℓ be the level-ℓ vertex just before $x_{\ell-1}$ on $[u,v]$ starting from u. By Lemma 27 and the fact that the portion $[x_\ell, x'_\ell]$ never goes to level $i < \ell$, we have $d_R(x_\ell, x'_\ell) \leq f(n)\max(32, 4\log_2 f(n)) - 1$. Therefore, $d_R(x'_\ell, y_\ell) \leq d_R(x_\ell, y_\ell) + d_R(x_\ell, x'_\ell) \leq 2f(n)\max(32, 4\log_2 f(n)) - 1$. Since $x_{\ell-1}$ and $y_{\ell-1}$ are the parents of x'_ℓ and y_ℓ respectively, by Lemma 16 we have $d_R(x_{\ell-1}, y_{\ell-1}) \leq f(n)\max(32, 4\log_2 f(n))$. Our claim holds for level $\ell - 1$. By induction, our claim stands.

We thus have $d(x_\ell, y_\ell) \leq 2\log_2(d_R(x_\ell, y_\ell)) + 2 \leq \max(32, 4\log_2 f(n))$, for all $\ell_0^{[]} \leq \ell \leq \ell_u$. For any vertex x between x_ℓ and x'_ℓ on $[u,v]$ (note that x may be at a level $\ell' \geq \ell$), by Lemma 27 we have $d(x, y_\ell) \leq d(x, x_\ell) + d(x_\ell, y_\ell) \leq d(x'_\ell, x_\ell) + d(x_\ell, y_\ell) \leq 2\max(32, 4\log_2 f(n))$. Hence all such vertices x are within distance $2\max(32, 4\log_2 f(n))$ from vertex y_ℓ in $\langle u,v\rangle$, and vice versa.

Similarly, we can define z_ℓ to be the the first level-ℓ vertex on geodesic $[v,u]$ starting from v, and w_ℓ to be the first level-ℓ vertex on geodesic $\langle v,u\rangle$ starting from v, for all $\ell_0^{[]} \leq \ell \leq \ell_v$. By a symmetric argument, we can show that w_ℓ are within distance $2\max(32, 4\log_2 f(n))$ from all vertices in the segment of $[u,v]$ from $z_{\ell-1}$ to z_ℓ for $\ell > \ell_0^{[]}$, and $d(z_{\ell_0^{[]}}, w_{\ell_0^{[]}}) \leq \max(32, 4\log_2 f(n))$.

The only portion left to argue is from $x_{\ell_0^{[]}}$ to $z_{\ell_0^{[]}}$ in geodesic $[u,v]$, and from $y_{\ell_0^{[]}}$ to $w_{\ell_0^{[]}}$ in geodesic $\langle u,v\rangle$. By Lemma 27 and the definition of $\ell_0^{[]}$, we know that $d(x_{\ell_0^{[]}}, z_{\ell_0^{[]}}) \leq \max(32, 4\log_2 f(n))$. Therefore, all vertices in the segment from $x_{\ell_0^{[]}}$ to $z_{\ell_0^{[]}}$ in geodesic $[u,v]$ are within $2\max(32, 4\log_2 f(n))$ to both $y_{\ell_0^{[]}}$ and $w_{\ell_0^{[]}}$. Now, for any vertex x in the segment from $y_{\ell_0^{[]}}$ to $w_{\ell_0^{[]}}$ in geodesic $\langle u,v\rangle$, we need to bound the distance from x to $[u,v]$. Since $d_R(y_{\ell_0^{[]}}, w_{\ell_0^{[]}}) \leq d_R(y_{\ell_0^{[]}}, x_{\ell_0^{[]}}) + d_R(x_{\ell_0^{[]}}, z_{\ell_0^{[]}}) + d_R(z_{\ell_0^{[]}}, w_{\ell_0^{[]}}) \leq$

$3f(n)\max(32, 4\log_2 f(n))$, we have $d(y_{\ell_0^{[]}}, w_{\ell_0^{[]}}) \leq 2\log_2 d_R(y_{\ell_0^{[]}}, w_{\ell_0^{[]}}) + 2 \leq \max(32, 4\log_2 f(n))$. Thus, for any vertex x in the segment from $y_{\ell_0^{[]}}$ to $w_{\ell_0^{[]}}$ in geodesic $\langle u, v\rangle$, x can reach either $x_{\ell_0^{[]}}$ and $z_{\ell_0^{[]}}$ in at most $2\max(32, 4\log_2 f(n))$ hops. Therefore, x is within distance $2\max(32, 4\log_2 f(n))$ from $[u, v]$. □

Proof of Theorem 3. We use Rips condition here. For any u, v, w, by Lemma 28, we have $[u, v] \subseteq B(\langle u, v\rangle, 2\max(32, 4\log_2 f(n)))$. By Lemma 24, $\langle u, v\rangle \subseteq B(\langle u, w\rangle \cup \langle v, w\rangle, 3)$. Again by Lemma 28, $\langle u, w\rangle \cup \langle v, w\rangle \subseteq B([u, w] \cup [v, w], 2\max(32, 4\log_2 f(n)))$. Combining these together, we have $[u, v] \subseteq B([u, w] \cup [v, w], 4\max(32, 4\log_2 f(n)) + 3)$. Therefore, since δ and δ_{Rips} differ within a constant factor, we have $\delta(RT(k, f)) \leq c\log f(n)$ for some constant c. □

4.3.5 Proof of Theorem 4

We first analyze the δ-hyperbolicity of $RRT(k, e^{-\alpha d_R(u,v)})$.

Proof of Theorem 4, part 1. $e^{\alpha} \leq \rho = \sum_{v \in V\setminus\{u\}} e^{-d_R(u,v)\alpha} \leq 2\sum_{i=1}^{+\infty} e^{i\alpha}$. Therefore, $\rho = \Theta(1)$. A vertex u on the leaves of $RT(k)$ has a long-range edge with ring distance greater than k with probability $\Theta(e^{-k\alpha})/\rho$. Let $k = \frac{2}{\alpha}\log n$, we know that a vertex has a long-range edge of ring distance greater than $\frac{2}{\alpha}\log n$ with probability $\Theta(1/n^2) = o(1/n)$.

Therefore, with probability $1 - o(1)$, long-range edges never exceed ring distance $\frac{2}{\alpha}\log n$. From Theorem 3 it follows that $\delta(RRT(k, e^{-\alpha d_R(u,v)})) = O(\log\log n)$, for any $\alpha > 0$. □

For the case of $\delta(RRT(k, d_R(u, v)^{-\alpha}))$, we first look at a lemma about the effect of long-range edges with ring distance $\Omega(n^c)$ on δ-hyperbolicity of ringed trees.

Lemma 29. *If we add an edge between u and v on the outermost ring of a k level ringed tree with $d_R(u, v) \geq c'n^c$, for some constants c and c', then the resulted graph G (possibly with other edges on the outermost ring) has $\delta(G) = \Omega(\log n)$.*

Proof. Let w be a node of lowest layer number on $\langle u, v\rangle$. By Lemma 25 and the structure of canonical geodesic, $d(w, u) \geq (2\log_2 c'n^c - 3)/2 \geq c\log_2 n + \log_2 c' - 3/2$. We consider the midpoint x of $[u, w]$. For any point y in $[v, w]$, by considering the canonical geodesic $\langle x, y\rangle$, we know that $d(x, y) \geq (c\log_2 n + \log_2 c' - 3/2)/2 - 3$. Therefore $\Delta(u, v, w)$ is at best $((c\log_2 n + \log_2 c' - 3/2)/2 - 3)$-slim, and thus $\delta(G) = \delta_{Rips}(G) = \Omega(\log n)$. □

A probabilistic version comes naturally as the following corollary.

Corollary 30. *For a random graph G formed by linking edges on leaves of a ringed tree. if for some constant c with $0 < c < 1$, with high probability there exists an edge linking some u and v with $d_R(u, v) = \Theta(n^c)$, then with high probability $\delta(G) = \Theta(\log n)$.*

Proof. Diameter of ringed tree gives $O(\log n)$ upper bound. Lemma 29 gives $\Omega(\log n)$ lower bound. □

We can now estimate the δ-hyperbolicity of $RRT(k, d_R(u, v)^{-\alpha})$.

Proof of Theorem 4, part 2. Note that in a ringed tree with n vertices, at least $n/2$ of them are leaves. For a constant c with $0 < c < 1$ and a fixed vertex u, the probability that the long-range edge (u, v) has $d_R(u, v) \leq (n/4)^c$ (we say that it is good) is $p = 2\rho^{-1}\sum_{d=1}^{(n/4)^c} d^{-\alpha}$, where $\rho = \Theta(\sum_{d=1}^{n} d^{-\alpha})$.

For $0 < \alpha < 1$, $p = O(n^{(1-\alpha)(c-1)}) = o(1)$. For $\alpha = 1$, $p = c + o(1)$. In these two cases, all edges are good with probability at most $(c + o(1))^{(n/2)} = o(1)$.

For $\alpha > 1$, $\rho = O(1)$. We take $q = 1 - p$. We have $q = O(n^{c(1-\alpha)})$. By picking $c = \min(1, \frac{1}{2(\alpha-1)})$, we have $q = O(n^{-1/2})$. All edges are good with probability $(1-q)^{n/2} = O(e^{-\sqrt{n}}) = o(1)$.

In all three cases, by Corollary 30, with probability $1 - o(1)$, $\delta(RRT(k, d_R(u,v)^{-\alpha})) = \Theta(\log n)$. □

Finally, we estimate the δ-hyperbolicity of $RRT(k, 2^{-\alpha h(u,v)})$.

Proof of Theorem 4, part 3. We note $n_L = (n+1)/2$ the number of leaves. Fix a leaf u. There are 2^{h-1} leaves v such that $h(u,v) = h$. Therefore $\rho = \sum_{h=1}^{\log_2 n_L} 2^{h-1-\alpha h}$. For a constant c with $0 < c < 1$, let $p(c) = \rho^{-1} \sum_{h=1}^{c \log_2 n_L} 2^{h-1-\alpha h}$ be the probability that u never links to any v with $h(u,v) \geq c \log_2 n$. We have $\rho = 2^{-\alpha} n_L^{1-\alpha}/(1 - 2^{1-\alpha})$ for $\alpha \neq 1$, and $\rho = \frac{1}{2} \log_2 n_L$ for $\alpha = 1$. For $\alpha \neq 1$, $p(c) = n_L^{-(1-\alpha)(1-c)}$. For $\alpha = 1$, $p(c) = c$.

For the case $\alpha \leq 1$, $p(1/2) = O(1)$. Therefore with probability $(1 - p(1/2))^{n_L} = o(1)$ there exists some u, v linked together with $h(u,v) \geq \frac{1}{2} \log_2 n$. For $\alpha > 1$, we take constant $c_0 = \min(1, \frac{1}{2(\alpha-1)})$, and $p(c_0) = n_L^{-1/2}$. Therefore with probability $(1 - p(c_0))^{n_L} = O(e^{-\sqrt{n_L}}) = o(1)$ linked together with $h(u,v) \geq c_0 \log_2 n_L$. In any cases, with $1 - o(1)$ probability, there exists u, v linked together by long-range edge with $h(u,v) \geq c \log_2 n_L$ for some constant c. We notice that this occurs uniformly through all edges.

Given u, v with $h(u,v) = h$, $d_R(u,v) < 2^{h/2}$ with probability at most $2^h 2^{1-2h} = 2^{1-h}$ by simply counting pairs within ring distance $2^{h/2}$. With $h(u,v) \geq c \log_2 n_L, c > 0$, we know that with probability $1 - 2n_L^{-c} = 1 - o(1)$ we have $d_R(u,v) \geq 2^{h/2} = n_L^{c/2}$. Combining with the previous analysis, we prove that with $1 - o(1)$ probability, there exists u, v linked together by long-range edge with $d_R(u,v) \geq n_L^{c/2}$ for some constant $c > 0$. By Corollary 30 and by $n_L > n/2$, with probability $1 - o(1)$ we have $\delta(RRT(k, 2^{-\alpha h(u,v)})) = \Theta(\log n)$. □

4.3.6 Proof of Theorem 5

We can order a binary tree to give it a ring distance. We will suppose that such a distance is defined hereinafter. We begin with a counter part of Lemma 29 in binary tree.

Lemma 31. *If there is an edge between two leaves u, v of a binary tree of size n with distance to lowest common ancestor $h(u,v) = c_1 \log_2 n + c_2$ for some constant $c_1 > 0, c_2 > 0$, then the resulted graph G (possibly with other edges on the outermost ring) has $\delta(G) = \Omega(\log n)$.*

Proof. Consider w the lowest common ancestor of u, v, and x the midpoint of $[u, w]$. We have $d(x, w) = d(w, u)/2 = h(u,v)/2$. For any y in $[w, v]$, $d(x, y) \geq h(u,v)/2$, as the only path in the tree from y to x always passes by w, and we need to climb $h(u,v)/2$ levels if we use links on leafs. Therefore $\Delta(u, v, w)$ is at best $h(u,v)/2 = \frac{1}{2}(c_1 \log_2 n + c_2)$, and we conclude that $\delta(G) = \Omega(\log n)$. □

Corollary 32. *For a random graph G formed by linking edges on leaves of a binary tree. if for some constant c with $0 < c < 1$, with high probability there exists an edge linking some u and v with $h(u,v) = \Theta(\log n)$, then with high probability $\delta(G) = \Theta(\log n)$.*

Proof. Diameter of binary tree gives $O(\log n)$ upper bound. Lemma 31 gives $\Omega(\log n)$ lower bound. □

Proof of Theorem 5. For $RBT(k, e^{-\alpha d_R(u,v)})$, the height of the whole tree is $h = \lfloor \log_2 n \rfloor$. There are $\Theta(\sqrt{n})$ subtrees of height $h/2$, with the root at the level $h/2$. For every neighboring such subtrees, the rightmost leaf u on the left subtree and the leftmost leaf v on the right subtree verifies $d_R(u, v) = 1$, $h(u, v) \geq h/2$. For each leaf, $\rho = O(1)$. Therefore, for u, v with $d_R(u, v) = 1$, there is an extra edge between u, v with constant probability $e^{-\alpha}\rho^{-1} > 0$. As there are $\sqrt{n}$ such pairs, with probability $1 - (1 - e^{-\alpha}\rho^{-1})^{\sqrt{n}} = 1 - o(1)$, there is a pair of leaves u, v linked by an extra edge with $d_R(u, v) = 1$, $h(u, v) \geq h/2 = \Theta(\log n)$. By Corollary 32, with probability $1 - o(1)$, $\delta(RBT(k, e^{-\alpha d_R(u,v)})) = \Theta(\log n)$.

For $RBT(k, d_R(u, v)^{-\alpha})$ and $RBT(k, 2^{-\alpha h(u,v)})$, using the same analysis in the proof of Theorem 4, we know that for some constant $c > 0$, with $1 - o(1)$ probability, there is an extra edge between u, v with $d_R(u, v) = \Omega(n^c)$. We have $h(u, v) = \Omega(\log n)$ because a subtree of height h spans a ring distance at most 2^h. By Corollary 32, we have $\delta(RBT(k, d_R(u, v)^{-\alpha})) = \delta(RBT(k, 2^{-\alpha h(u,v)})) = \Theta(\log n)$. □

4.4 Extensions of random ringed tree model

We will now discuss some extensions of the random ringed tree (RRT) model, and show that our results still hold for these extensions, thus extending its expressivity.

We start from some observations in the proof of Theorem 4. In this proof, the upper bound of δ-hyperbolicity is given by Theorem 3, and the lower bound is given by Corollary 30. In the statement of Theorem 3, by the definition of $RT(k, f)$, only a uniform bound of ring distance $d_R(u, v)$ for each long-range edge (u, v) is considered. In the statement of Corollary 30, the only quantity concerning a long-range edge (u, v) is also the ringed distance between u and v, and to apply this corollary, we only need to show that a long-range edge (u, v) with $d_R(u, v) = \Theta(n^c)$ for some constant c exists with high probability. Therefore, the proof of Theorem 4 relies only on the ring distances of long-range edges.

To extend the RRT model while keeping similar properties on δ-hyperbolicity, we only need to show that Theorem 3 and Corollary 30 are still applicable in these extensions. We will here discuss two extensions on choosing long-range edges.

A constant number of long-range edges for each node. In the original RRT model, each node only have one long-range edge connecting to other nodes. We can extend the model to allow each node to have a constant number of long-range edges connecting to a constant number of other nodes. In this extension, Theorem 4 still holds, since Theorem 3 is not concerned by the number of long-range edges, and Corollary 30 is still applicable as the required probability only increases with extra long-range edges.

Independent long-range edges. In the original RRT model, we choose exactly one long-range edge for each node. A variant of the model is that each node u can choose edge (u, v) independently from other edges (u, v'), with the same probability as in the original model such that on expectation u connect out with one long-range edge. In this variant, Theorem 4 still holds. The reason is that in expectation, at least a constant fraction of nodes issue only one edge, and the computation for applying Corollary 30 stays similar. It is clear that the application of Theorem 3 stays valid.

In the two variants discussed above, we can see that Theorem 4 still applies, and we have exactly the same property on δ-hyperbolicity of these variants. We can also combine these two variants, and it is clear that our results are still valid.

5 Discussions and open problems

Perhaps the most obvious extension of our results is to close the gap in the bounds on the hyperbolicity in the low-dimensional small-world model when γ is at the "sweetspot," as well as extending the results for large γ to dimensions $d \geq 2$. Also of interest is characterizing in more detail the hyperbolicity properties of other random graph models, in particular those that have substantial heavy-tailed properties. Finally, exact computation of δ by its definition takes $O(n^4)$ time, which is not scalable to large graphs, and thus the design of more efficient exact or approximation algorithms would be of interest.

From a broader perspective, however, our results suggest that δ is a measure of tree-like-ness that can be quite sensitive to noise in graphs, and in particular to randomness as it is implemented in common network generative models. Moreover, our results for the δ hyperbolicity of rewired trees versus rewired low-δ tree-like metrics suggest that, while quite appropriate for continuous negatively-curved manifolds, the usual definition of δ may be somewhat less useful for discrete graphs. Thus, it would be of interest to address questions such as: does there exist a measure other than Gromov's δ that is more appropriate for graph-based data or more robust to noise/randomness as it is used in popular network generation models; is it possible to incorporate in a meaningful way nontrivial randomness in other low δ-hyperbolicity graph families; and can we construct non-trivial random graph families that contain as much randomness as possible while having low δ-hyperbolicity comparing to graph diameter?

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